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A THEORY OF THE FILAMENT TEMPERATURE DISTRIBUTION OF THE TUNGSTEN VACUUM-LAMP, WITH SPECIAL REFERENCE TO OPTICAL PYROMETRY

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CONTENTS

The paper gives a general theory designed mainly for application to the two types of vacuum lamps commonly used in optical pyrometry: namely, the heavy-current strip lamps which serve as standards for calibration and the fine filament lamps embodied in the pyrometers themselves. Simplifying assumptions are made regarding the properties of tungsten, namely, that resistivity and emissivity both vary directly as the absolute temperature, while thermal conductivity remains constant. These allow a complete solution of the main differential equation governing temperature distribution. A comparison of results obtained in this way with a few computations, specially undertaken without resort to the resistivity and emissivity assumptions, shows that the simplifications suffice for all practical purposes. Further, the general agreement between the calculated and observed characteristics of lamps suggests that no great error results from the assumption of constant thermal conductivity. However, other simplifications are also briefly considered.

The problems treated on the basis indicated are as follows: the distribution of temperature along a filament of known dimensions, carrying a constant current and with its ends at a constant temperature; the voltage on, and resistance of, such a filament; the temperature coefficients of calibration, i.e. the change in maximum temperature with change in surrounding temperature with the filament run at constant amps, volts or ohms, as the case may be; the modifications of the above-mentioned characteristics due to appreciable gradients in the leads, and the asymmetry of distribution induced by the Peltier and Thomson effects. These subjects are treated generally for short filaments and also for the simpler case of infinitely long filaments to which, in fact, many lamps tend to conform. In addition to these matters, which are all concerned with the steady state of temperature distribution, the approach to that state is considered under conditions giving an estimate of the maximum speed of response attainable in practice.

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Numerical examples of the formulae developed are given for typical lamps and some of these are compared with other calculations and with observations. For pyrometer lamps a table is given showing how their main characteristics can be estimated, with fair accuracy, merely from a knowledge of the filament dimensions, or, in the case of existing lamps, by means of a simple test. Suggestions are also made for possible modifications in the design of standard strip lamps.

1·1. Introduction

The vacuum lamp, with tungsten filament, is used for precise measurement in several fields of investigation, notably in photometry, colour pyrometry, and brightness, or monochromatic, pyrometry. Perhaps the highest demands are made on it in brightness pyrometry, where it provides the means for realizing the International Temperature Scale from the melting point of gold (1063 °C) upwards. The brightness pyrometer is also much used below 1063 °C—in fact down to 650 °C or near the visible limit—and in part of its wide range a precision of 1 °C, or better, is sometimes aimed at. This paper, while giving a theory applicable generally to tungsten vacuum-lamps, is concerned mainly with problems arising out of the use of such lamps in brightness pyrometry.

In the best form of brightness pyrometer, that of the disappearing-filament type, the brightness, for a certain wavelength, of the hottest part of the filament is matched against that of the object, and this brightness is reproduced in terms of some characteristic of the lamp: for example, the current through the filament, the overall voltage on the lamp, or its overall resistance. It is, of course, of first importance that the lamp should be stable under all conditions met in practice, including particularly the unavoidable changes of atmospheric temperature to which it is subjected. The elimination of the effect of such changes on the calibration of the lamp is best secured by employing a filament very long in proportion to its cross-section and dealing with the current characteristic, which is not directly affected by the resistance of the leads. This is the universal practice in laboratory work of the highest precision. The requirements of industrial pyrometry are less exacting and lamps departing from the high standards of the laboratory, and a characteristic other than current, may with advantage be employed to gain other desiderata, particularly in relation to the portability of the instrument and the interchangeability of lamps.

In addition to the lamp embodied as an integral part of the pyrometer, special lamps, calibrated in terms of the International Temperature Scale, are employed as standards for calibrating pyrometers and for international comparison of temperature scales. To provide a field of view sufficiently large for the purpose, the filament consists of tungsten strip of considerable width. The highest standards of precision and stability are demanded of such lamps, which call for special design.

The various problems connected with pyrometer and standard lamps can be tackled on a purely empirical basis, and much valuable work of this kind has been done; but there is obvious advantage in developing an appropriate theory. For example, it would be useful to be able to calculate the temperature distribution, current, voltage and resistance of a filament of known dimensions and physical constants, in terms of the temperature of its centre, ends and surroundings.

The theories so far advanced have not dealt specifically with all such points. The most noteworthy of the earlier contributors appear to have been Worthing, Langmuir and his associates, and Ribaud. Worthing (1922, 1914*a*) has given separate treatments of the temperature distribution in a filament in regions near its hottest part, and for an infinite filament in the region of a cooling junction; Langmuir, MacLane & Blodgett (1930) have given a general theory of wide application making use of the empirical laws for the properties of tungsten and nickel, but which would apparently involve laborious calculation in dealing with the problems indicated above; Ribaud & Nikitine (1927) have, in effect, extrapolated the theory of Worthing (1922) to the cold ends of the filament and so obtained a simple expression for the temperature distribution along its whole length (see also the appendix to the present paper). The several works referred to above were carried out long before the present work was planned; recently papers have appeared dealing with some of the more general aspects of the heat conduction in a thin rod electrically heated *in vacuo* (Jain & Krishnan, 1954*a*, *b*, *c*, *d*; 1955*a*, *b*) but the scope of these did not include any new attempt to obtain a complete solution, in terms of known functions, of the fundamental differential equations involved; for short filaments the solution of Ribaud & Nikitine was used.

Our approach to the problems connected with the vacuum lamp has been somewhat different from any of the above. Confining attention to the tungsten-filament lamp, which is the most widely used type, we have examined critically the data for the properties of tungsten and have ventured to make certain simplifying assumptions concerning them which permit of a complete solution of the main differential equation governing the distribution of temperature along the filament. The validity of our more important assumptions has, with the co-operation of the National Physical Laboratory, been tested in a few specimen cases by comparing the results yielded by them with those not involving any such simplifications. The other matters dealt with include a brief consideration of further approximations; the modifications of distribution due to the Peltier and Thomson effects and to the material and dimensions of the leads; the effect of changes of air temperature on the lamp calibration. In addition we have given some attention to the approach to the steady state, a matter which is of importance in relation to standard strip lamps.

1·2. The physical constants of tungsten

The physical properties of tungsten which in the main determine the equilibrium distribution of temperature in a filament, are the emissivity and resistivity, if the filament is infinitely long; and, in addition, the thermal conductivity, if the filament is short. These properties give a temperature distribution independent of the direction of the current. Further there are the Peltier and Thomson effects, tending to disturb the symmetry of distribution.

The more important properties of tungsten, which have been named above, are liable to be affected by the purity and physical condition of the metal; but in a well-aged filament they seem to be reasonably stable, as is witnessed by the use of such filaments as photometric and pyrometric standards. We deal below with the data for the three constants, their changes with temperature and their interrelations.

1·2·1. *Thermal conductivity*

A cursory examination of any collection of physical constants reveals that thermal conductivity is among the most difficult of these to determine. This is exemplified in table 1 relating to tungsten.

The differences shown are not surprising in view of the fact that the measurements have been made on fine wires and depend essentially on two difficult determinations, namely, the temperature gradient along the wire and the heat conducted by it. The last-mentioned is derived from the difference between two large quantities, namely, the heat generated by the current and that lost by radiation. Under these conditions precise determinations are hardly to be expected, especially of a second-order effect like the change of thermal conductivity with temperature.

TABLE 1. THERMAL CONDUCTIVITY OF TUNGSTEN (W CM/°C)

Estimated or extrapolated values in ( ).

authority	temperature (°K) 300	1000	1500	2500
Coolidge	1·47	—	—	—
Worthing (1914*b*)	—	(0·73)	0·98	1·48
Langmuir (1916*a*)	(1·14)	1·14	1·14	1·14
Worthing (1917)	—	(0·85)	1·01	1·25
Weber (1918)	1·60	—	—	—
Zwikker (1925)	—	(0·98)	1·12	1·38
Forsythe & Worthing (1925)	(0·63)	0·84	0·99	1·21
Kannuluik (1933)	1·65	—	—	—
Michels & Cox (1936)	1·66	—	—	—
Osborn (1941)	—	(1·19)	1·11	1·03
electron theory*	(1·23)	—	—	—

* The value shown is obtained by assuming Lorenz's function $k\rho/T$ to be $2{\cdot}32\times10^{-8}$, the average value for the pure metals at room temperature, and the resistivity at 300 °K to be $5{\cdot}64\times10^{-6}$.

Coming to the actual values, it may be mentioned that Forsythe & Worthing give the thermal conductivity as increasing in a nearly linear way for temperatures between 1000 and 2800 °K (only three of their values are entered in the table); these values and those of Worthing (1917) were in fact corrected from those of Worthing (1914*b*) on the basis of new determinations of a true temperature scale for tungsten. As a check on this variation we have extrapolated their values to 300 °K; it will be seen that the resultant figure is much lower than any in the table. The values attributed to Langmuir were recalculated by him from the observations of Worthing (1914*b*) on the basis of his own determinations of the total emissivity of tungsten (Langmuir 1916*b*), which differed considerably from those of the observers named. Langmuir thus arrives at a nearly constant value for the thermal conductivity, which does not differ greatly from the other figures at 300 °K. In view of the conflicting data and of the fact that a number of pure metals show only small change in value, either positive or negative, with rising temperature, we feel justified in assuming a constant value for the thermal conductivity.

1·2·2. *Resistivity and total emissivity*

The relation between these two quantities can be determined under conditions practically identical with those in a tungsten-filament lamp. Thus if a very long filament is heated electrically *in vacuo* and thin potential leads are applied to a central length of the filament, sensibly at a uniform temperature, the measurement of the current through, and the potential drop on, this length, gives both its resistance and the energy radiated from it. To obtain from these data the actual resistivity and total emissivity requires, in addition to details of dimensions, a determination of the true temperature of the filament. The establishment of the temperature relation of either resistivity or emissivity would serve to fix the other, but in both cases the measurement of filament temperature presents great difficulty. If it were possible to apply the technique of resistance thermometry to tungsten (which does not seem to have been attempted) the measurement of temperature could be made with considerable precision, since it would be that of the medium surrounding the bulb—such as, for example, the atmosphere inside a furnace, or a metal at its melting point. In this way Callendar (1899) was able to measure the resistance of platinum at the melting point of palladium (1825 °K) with a precision equivalent to 1 or 2 °C. However, the plan usually adopted is to measure by optical pyrometer the *brightness temperature* of the filament, i.e. that of a black body which directly matches the filament in brightness. This measurement would apply to a certain wavelength (generally about 0·65 μ) and from a knowledge of the spectral emissivity for this wavelength, determined by separate experiment, the *true temperature* of the filament would be obtained. The corrections from brightness to true temperature range from some 35° at a true temperature of 1000 °K to 225° at 2500 °K (see Ribaud 1931, table 67), and suggest the possibility of appreciable error. When a filament is partially obscured by absorptive material (such as a glass bulb) we designate as its *apparent temperature* that of a black body which matches the obscured filament in brightness. Thus apparent temperature is a function of the optical arrangement as well as of emissivity, and corrections for both must be made in converting to true temperature; e.g. a tungsten filament at a true temperature of 1400 °K would have a brightness temperature at 0·665 μ of 1330 °K, and an apparent temperature, as viewed from outside a clear glass envelope with 8 % absorption, of 1323 °K (1050 °C). There are other methods, in addition to that outlined above, for determining the relation of total emissivity to true temperature, but they all present great experimental difficulties.

Dealing first with the total radiation the energy radiated (W_T) may be taken to vary with the true temperature (T) expressed in absolute measure (°K) in either of the forms

$$W_T \propto T^{\gamma} \quad \text{or} \quad W_T \propto \epsilon_T T^4, \tag{1·2·1}$$

where ϵ_T is the total emissivity at temperature T. Forsythe & Worthing (1925) gave γ in the above formula as ranging from 5·35 at 1000 °K to 4·59 at 2500 °K. On the other hand, Langmuir (1916 *b*) gives a constant γ of 4·96. We have adopted the value of Langmuir since it accords with his constancy of thermal conductivity which we have already accepted. For the purpose of our approximation we take the value of 5 for γ, which is equivalent to making ϵ_T vary directly with T.

Coming to resistivity (ρ_T), if its relation to T is assumed to be of the form

$$\rho_T \propto T^{\beta}, \tag{1·2·2}$$

then Forsythe & Worthing give β as 1·20 from 1000 °K upwards, but steadily increasing to 1·3 at 300 °K. However, Langmuir gives a constant value of 1·24. For the purpose of our approximation we take a value of 1·00 for β. This represents perhaps the greatest departure from the published data for the three constants, but some justification for it may be sought in the relation of resistivity to the other two. Thus, if we consider the central part of an incandescent filament, most of the heat dissipated will be by radiation, so that here resistivity and emissivity predominate in determining temperature. On the other hand, resistivity and thermal conductivity become the more important factors towards the ends of the filament. It may be useful, therefore, to consider the relations between the two pairs of constants.

1·2·3. *Relation between resistivity and emissivity*

The following theoretical relation has been derived by Aschkinass (1905) for the 'total normal emissivity'

$$\epsilon_{nT} \propto (\rho T)^{\frac{1}{2}}. \tag{1·2·3}$$

It will be observed that our assumption of ρ varying directly with T would also make ϵ_{nT} so vary, which agrees with our assumption for emissivity. We are not, however, strictly concerned with ϵ_{nT} but with ϵ_{hT} or the 'hemispherical total emissivity'.* For this latter entity a theoretical expression has also been derived, the first term of which is of the form of (1·2·3) so that our assumption would still hold approximately.

1·2·4. *Relation between thermal conductivity and resistivity*

According to Lorenz, the thermal conductivity (k) and resistivity (ρ) of a pure metal are connected to the absolute temperature (T) by the relation

$$k\rho/T = \text{constant}.$$

This means that if k is constant, ρ would vary directly with T, which would make our assumptions, as to these two constants, mutually consistent. There is very little evidence of the validity of the Lorenz relation in the case of tungsten (though see Langmuir 1916*b*), but it is known to hold closely for all the pure metals tested in the region of 300 °K, while for several hundred degrees above 300 °K the limited data available show that it still holds approximately for a number of metals.

2. THEORY

The whole of this section, except for §2·3, is based on the assumptions set out in the preceding section, namely, of constant thermal conductivity, and of resistivity and emissivity varying directly with absolute temperature. In §2·3 the possibilities of less drastic simplifications are briefly explored.

The subjects treated are the distribution of temperature and resistance along the filament (both of finite and infinite length) under specified conditions: the modification of these induced by the dimensions and material of the leads, and by the Peltier and Thomson effects: the effect on calibration data of varying ambient temperature: the speed of response of filaments.

The main symbols used are as follows:

T true temperature at a point on the filament, in absolute measure (° K)
k_T thermal conductivity of filament at temperature T
ρ_T resistivity of filament at temperature T
ϵ_T emissivity or absorptivity of filament at temperature T
σ constant of total black-body radiation
ψ_T Thomson coefficient of filament at temperature T
Π_j Peltier coefficient between filament and lead at temperature T_j

* For definition of terms, and an authoritative discussion of emissivity, see Worthing (1941).

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A cross-sectional area of filament
p perimeter of cross-section
L semi-length of filament
A_1 cross-sectional area of a lead
k_1 thermal conductivity of a lead
l length of a lead
T_s temperature of surroundings
T_j temperature of filament end
T_m maximum temperature of finite filament with current c
T_∞ temperature of infinite filament

2·1. Finite filament: temperature distribution and resistance

2·1·1. *Temperature distribution*

The differential equation for the steady flow of heat across an element, dx in thickness and at a temperature T, of a filament is

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak_T\frac{\mathrm{d}T}{\mathrm{d}x}\right)=\frac{c^2\rho_T}{A}-\sigma p\epsilon_T(T^4-T_s^4); \qquad (2\cdot1\cdot1)$$

with our assumptions

$$\rho_T=\rho_0 T,\quad \epsilon_T=\epsilon_0 T,\quad k_T=k\quad(\rho_0,\epsilon_0,k\text{ constant}),$$

(2·1·1) becomes $$-Ak\,\mathrm{d}^2T/\mathrm{d}x^2=(c^2\rho_0/A+\sigma p\epsilon_0 T_s^4)\,T-\sigma p\epsilon_0 T^5$$

or $$Ak\,\mathrm{d}^2T/\mathrm{d}x^2=\sigma p\epsilon_0(T^5-BT),$$

where* $$B=T_s^4+c^2\rho_0/A\sigma p\epsilon_0.$$

Hence, upon integrating,

$$\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)^2=\frac{\sigma p\epsilon_0}{3Ak}(T^6-3BT^2+\text{constant}),$$

where the constant is determined by $\mathrm{d}T/\mathrm{d}x=0$ when $T=T_m$, T_m being the maximum temperature in the filament. The bi-cubic function of T yields an elliptic function on integration, a convenient treatment being as follows. Substitute $z=(T/T_m)^2$, so that $\mathrm{d}T/\mathrm{d}x=0$ when $z=1$, and put $Q=3B/T_m^4$; we then obtain

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2=\frac{4\sigma p\epsilon_0 T_m^4}{3Ak}z(1-z)(z^2+z+1-Q).$$

If we write $\lambda=\frac{1}{2}[\surd(4Q-3)-1]$, the roots of $z^2+z+1-Q=0$ are λ and $-(\lambda+1)$; we note that, in terms of our basic quantities,

$$\lambda=\frac{1}{2}\left\{\sqrt{\left[\frac{12}{T_m^4}\left(T_s^4+\frac{c^2\rho_0}{A\sigma p\epsilon_0}\right)-3\right]}-1\right\}.$$

If we then put $$\xi=\frac{1}{T_m^2}\sqrt{\left[\frac{Ak}{\sigma p\epsilon_0}\frac{3}{\lambda(\lambda+2)}\right]}$$

* We can also write $B=T_\infty^4$, where T_∞ is the temperature which would be maintained by the current c if the filament were infinite; see equation (2·2·1).

the differential equation becomes

$$\left(\frac{dz}{dx}\right)^2 = \frac{4}{\xi^2\lambda(\lambda+2)} z(1-z)(\lambda-z)(\lambda+1+z). \tag{2·1·2}$$

This can be reduced to a Legendre form

$$x = \xi\int(1-\kappa_1^2\sin^2\phi)^{-\frac{1}{2}}\,d\phi = \xi F(\phi,\kappa_1) + \text{constant},$$

or, if the origin is at the point of maximum temperature,

$$x = \xi F(\phi, \kappa_1), \tag{2·1·3}$$

where the modulus and amplitude are given by

$$\left.\begin{aligned}\kappa_1^2 &= (2\lambda+1)/(\lambda^2+2\lambda),\\ \sin^2\phi &= \lambda(1-z)/(\lambda-z).\end{aligned}\right\} \tag{2·1·4}$$

It can be shown that the condition $\lambda > 1$ is always satisfied which makes $\kappa_1^2 < 1$ as required. The above equations are fundamental throughout this paper and have been used for numerical computation with Legendre's *Tables*, in conjunction with the corrections given by Kaplan (1948) (see also Fletcher 1948).

For any filament of given cross-section, there are four variables to be taken into account, namely, current (c), length ($2L$), temperatures of centre and ends (T_m and T_j for a symmetrical distribution). Given any three of these, the fourth may be obtained from (2·1·3), generally by trial. But the effect of leads has also to be considered. If these are so heavy as to have no appreciable temperature gradient, T_j becomes equal to T_s (temperature of surroundings) and can be treated as constant. Equation (2·1·3) then gives the temperature distribution for any two of the remaining variables, ignoring the small disturbance due to the Thomson effect. With appreciable gradient in the leads, special treatment is required to determine T_j, as is shown in §2·4 below, but (2·1·3) would still apply to the distribution in the filament. It may be added that (2·1·3) can be inverted to yield

$$T = \frac{T_m\sqrt{\lambda}\,\mathrm{cn}\,(x/\xi)}{\sqrt{[\lambda-\mathrm{sn}^2\,(x/\xi)]}}, \tag{2·1·5}$$

in which case use may be made of the *Smithsonian elliptic functions tables* (Spenceley & Spenceley 1947).

2·1·2. *Resistance*

To obtain an expression for the resistance r of any section of the filament we use

$$A\,dr = \rho_T\,dx,$$

so that, writing

$$\eta = \frac{\rho_0}{T_m}\sqrt{\left[\frac{k}{A\sigma\rho\epsilon_0}\frac{3}{2\lambda+1}\right]}$$

and substituting in (2·1·2),

$$\left(\frac{dz}{dr}\right)^2 = \frac{4}{\eta^2(2\lambda+1)}(1-z)(\lambda-z)(\lambda+1+z) \tag{2·1·6}$$

which, with origin at $T = T_m$, yields the solution

$$r = \eta F(\theta, \kappa_2), \tag{2·1·7}$$

where the modulus and amplitude are given by

$$\kappa_2^2 = (\lambda+2)/(2\lambda+1),$$

$$\sin^2\theta = \frac{(2\lambda+1)(1-z)}{(\lambda+2)(\lambda-z)}.$$

We note that these are related to values in (2·1·4) as follows:

$$\sin\theta = \kappa_1 \sin\phi \quad \text{and} \quad \lambda\kappa_1^2\kappa_2^2 = 1. \qquad (2·1·8)$$

As before, (2·1·7) may be inverted to give

$$T = T_m\sqrt{[\lambda-(\lambda-1)/\mathrm{dn}^2(r/\eta)]}. \qquad (2·1·9)$$

(2·1·7) or (2·1·9) gives the relation between temperature and resistance. If it is desired to relate length with resistance we eliminate $z = (T/T_m)^2$ between (2·1·5) and (2·1·9) and so obtain, on simplification,

$$\kappa_1 \,\mathrm{sn}\,(x/\xi, \kappa_1) = \mathrm{sn}\,(r/\eta, \kappa_2). \qquad (2·1·10)$$

2·2. Infinite filament: temperature distribution and resistance

The preceding section (§2·1) covers any filament, however short, but many pyrometer filaments are so long, relative to cross-section, as to differ little from those infinitely long. Consequently it is useful to develop the relations for these latter. We first give the relation between current and temperature

$$c^2\rho_\infty/A = \sigma p\epsilon_\infty(T_\infty^4 - T_s^4),$$

where $\rho_\infty = \rho_{T\infty}$, $\epsilon_\infty = \epsilon_{T\infty}$, which, on applying our assumptions, becomes

$$c^2 = (\sigma p\epsilon_0 A/\rho_0)(T_\infty^4 - T_s^4); \qquad (2·2·1)$$

T_s^4 is always small relative to T_∞^4 so that $c \propto T_\infty^2$ approximately. As a check we note that this would give the ratio of currents for 2000 and 1000 °K (about the limits met in pyrometry) to be 4. The published data of Forsythe & Worthing (1925) would make this ratio 3·97, which tends to confirm the validity of our assumptions.

We now consider a finite filament sufficiently long that in its central section the temperature is uniform and indistinguishable from that of a similar infinitely long filament carrying the same current. The resistance and temperature distribution near a cold end of the filament can then be readily derived from the formulae in §2·1. Thus if we substitute for the current the value (2·2·1) we get

$$\lambda = 1 \quad \text{and} \quad \xi = \mu,$$

where

$$\mu = \frac{1}{T_\infty^2}\sqrt{\frac{Ak}{\sigma p\epsilon_0}}.$$

With $z = (T/T_\infty)^2$, (2·1·2) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2}{\mu\sqrt{3}}(1-z)\,[z(2+z)]^{\frac{1}{2}}$$

of which the solution is

$$x = \mu \tanh^{-1}\sqrt{\frac{3z}{2+z}} + \text{constant}.$$

If the temperature at one end of the filament is T_j ($z_j = T_j^2/T_\infty^2$), and x is measured from this end,

$$x = \frac{1}{T_\infty^2}\left(\frac{Ak}{\sigma p \epsilon_0}\right)^{\frac{1}{2}}\left\{\tanh^{-1}\sqrt{\frac{3z}{2+z}} - \tanh^{-1}\sqrt{\frac{3z_j}{2+z_j}}\right\}. \tag{2·2·2}$$

The corresponding value of the resistance measured from the end is

$$r = \frac{\rho_0}{T_\infty}\left(\frac{k}{A\sigma p \epsilon_0}\right)^{\frac{1}{2}}\left\{\tanh^{-1}\sqrt{\frac{2+z}{3}} - \tanh^{-1}\sqrt{\frac{2+z_j}{3}}\right\}. \tag{2·2·3}$$

If desired, a relation may be obtained between r and x corresponding with (2·1·10).

A useful modification of (2·2·2) is designed to give the distance x_Δ at which the temperature reaches a small amount Δ short of its final value T_∞. This enables a length of finite filament to be chosen so as to approximate, for any fixed maximum temperature, to an infinite filament (cf. Ribaud 1931, table 48). It can be shown that x_Δ is given closely by

$$x_\Delta = \frac{1}{T_\infty^2}\left(\frac{Ak}{\sigma p \epsilon_0}\right)^{\frac{1}{2}}\left\{\tfrac{1}{2}\ln\frac{3T_\infty}{\Delta} - \tanh^{-1}\sqrt{\frac{3z_j}{2+z_j}}\right\}; \tag{2·2·4}$$

if, further, z_j is small, the last term in the bracket can be replaced by $(T_j/T_\infty)\sqrt{\frac{3}{2}}$. In the corresponding expression for r_Δ the first term in the bracket of (2·2·3) can be written $\frac{1}{2}\ln(6T_\infty/\Delta)$.

2·3. Finite filament: closer approximation for temperature distribution

So far the treatment has been based on the simplifying assumptions $\rho_T = \rho_0 T$, $\epsilon_T = \epsilon_0 T$, k_T constant. In this section the possibilities of less drastic simplifications are briefly considered.

First, we assume k_T to remain constant: then if the resistivity and emissivity can be expressed as integrable functions of T, the first stage of integration obviously presents no difficulty, but the final stage will not generally yield a solution in terms of known functions and so may involve a troublesome numerical process. Taking, for example, resistivity, where our assumptions have involved the greatest departure from published data, the following relation would no doubt give a closer approximation to accepted values

$$\rho_T = \rho_0 T(1+\alpha_1 T + \alpha_2 T^2).$$

The presence of the second term in the bracket, yielding a T^3 term on integration, would preclude the possibility of a discrete solution on final integration. If, however, the second term were omitted, and the relation taken as

$$\rho_T = \rho_0 T(1+\alpha T^2), \tag{2·3·1}$$

though the matching with accepted data would be less close, the first integration of (2·1·1) would yield, as before, a bi-cubic expression, reducible to a Legendre F-function. The assumption (2·3·1) with k_T and ϵ_T as before, could therefore be taken as the basis for a second approximation. We have in fact worked out the complete solutions in this case, corresponding to all those in §2·1 and, further, have considered how the two parameters ρ_0 and α could be matched to the best advantage. However, on applying these two approximations to a specimen case, we find that the first approximation ($\rho_T = \rho_0 T$) yields results so close to the calculations of the National Physical Laboratory, free from any assumptions, as practically to render the second approximation unnecessary (see figures for N.P.L. and R. & S. in table 2). Accordingly the algebra for the second approximation is not reproduced here, though two specimen results derived from it will be found in the table.

With regard to thermal conductivity, although its variation is small and not known with any certainty, it may not be out of place to consider how a simple form of variation could be treated. Let us assume

$$k_T = k_0(1+\nu T);$$

then the left side of equation (2·1·1) becomes

$$-Ak_0\frac{\mathrm{d}^2T}{\mathrm{d}x^2}-Ak_0\nu\left\{T\frac{\mathrm{d}^2T}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)^2\right\}.$$

If we first neglect as small the term involving ν, and take the variations of resistivity and emissivity to be as originally assumed, or ρ_T of the form (2·3·1), then the first integration of (2·1·1) will give an expression for $(\mathrm{d}T/\mathrm{d}x)^2$ in terms of rising powers of T. The approximate expressions for $\mathrm{d}^2T/\mathrm{d}x^2$ and $(\mathrm{d}T/\mathrm{d}x)^2$ can now be substituted in the ν term, and on integration a second approximation obtained for $(\mathrm{d}T/\mathrm{d}x)^2$. The 'feed-back' process could be repeated, if necessary, and the final expression for $(\mathrm{d}T/\mathrm{d}x)^2$ would be in rising powers of T, and have to be solved numerically.

2·4. Conduction in the leads: Peltier effect

2·4·1. *Lead conduction and temperature distribution*

In order to get a complete picture of the temperature distribution in any lamp, it is necessary to take account of the effect of the leads. Though no specifications have been laid down for the materials and dimensions of leads, there is a fairly uniform pattern, at any rate in this country, for the strip lamps used as standards. It is convenient to consider the various problems in relation to this particular design which is illustrated in figure 1, which shows two sections of the lamp at right angles to each other. Here the leads are of unequal length and are connected by the sealing-in wires to an indeterminate mass of material in the lamp base. To permit of treatment, we assume this set-up to consist of leads of uniform section and definite lengths connected to infinite sinks at the temperature of the surroundings, as indicated in figure 2. It will be understood that this simplification, with its uncertainty as to the effective length of a lead, or even as to the constancy of this length under varying conditions, must give results of the nature of estimates, rather than precise determinations.

The first problem is to calculate the temperature T_j of the junction between filament and lead, for given T_m and T_s. The condition for continuity of heat flow at this point is

$$A_1k_1\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_j = Ak\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_{jw}, \qquad (2·4·1)$$

where $(\mathrm{d}T/\mathrm{d}x)_j$ and $(\mathrm{d}T/\mathrm{d}x)_{jw}$ are the gradients at the junction in lead and filament, respectively, and A_1, A and k_1, k the respective cross-sections and thermal conductivities. The gradients and temperature distributions in filament and leads are governed by the general differential equation (2·1·1), and the formulae for the filament have been fully worked out. With a symmetrical distribution in the filament (of given length) and fixed values of T_m and

T_s we could obtain by trial from (2·1·3) the current which would give any chosen value of T_j; (2·1·2) would then give the corresponding value of $Ak(\mathrm{d}T/\mathrm{d}x)_{jw}$, and a set of values for the last-mentioned could then be plotted for a number of T_j values. For a lead, the solution of (2·1·1) would be required in terms of its length, current, and its two end temperatures T_s and T_j; taking a series of values for T_j with the corresponding currents (already found for the filament), a curve could be obtained for $A_1 k_1 (\mathrm{d}T/\mathrm{d}x)_j$ against T_j. The intersection of the two curves would give the required value of T_j.

In view of the uncertainty of the length of lead and the laborious procedure outlined it is worth considering whether a plausible approximation could be developed on simpler lines. It will be seen from equation (2·1·1) that the heat generated in the lead by the current

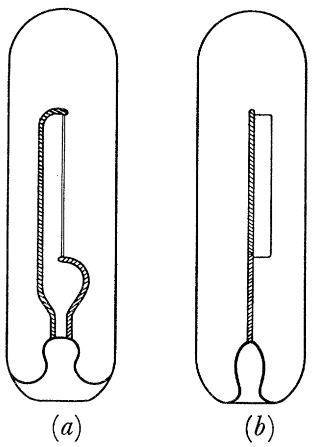

FIGURE 1. Sections of a standard strip lamp.

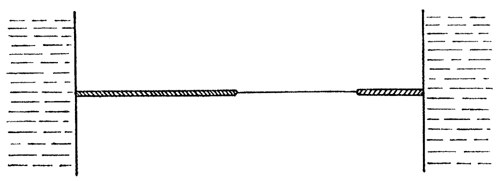

FIGURE 2. Simplified representation of a standard strip lamp.

tends to be compensated for by that lost in radiation from the lead, and if it is assumed that the two are equal* the result would be a constant gradient in the leads, thus making

$$(\mathrm{d}T/\mathrm{d}x)_j = (T_j - T_s)/l, \tag{2·4·2}$$

where l is the length of the lead. Coming to the filament we have, from (2·1·2),

$$\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_{jw} = T_m^3 \left\{\frac{\sigma p \epsilon_0}{3Ak}(1-z_j)(\lambda - z_j)(\lambda + 1 + z_j)\right\}^{\frac{1}{2}} \tag{2·4·3}$$

$$\simeq T_m^3 \left\{\frac{\sigma p \epsilon_0 \lambda(\lambda+1)}{3Ak}\right\}^{\frac{1}{2}} \left\{1 - \frac{(\lambda^2+\lambda+1)}{2\lambda(\lambda+1)} \frac{T_j^2}{T_m^2}\right\}, \tag{2·4·4}$$

the expansion being carried out only to the second term, since in practice $z_j = (T_j/T_m)^2$ is small enough to justify the omission of further terms. Many filaments will, however, approximate to one infinitely long, in which case $\lambda = 1$ and we have

$$\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_{jw} = T_\infty^3 \left\{\frac{2\sigma p \epsilon_0}{3Ak}\right\}^{\frac{1}{2}} \left\{1 - \frac{3T_j^2}{4T_\infty^2}\right\}. \tag{2·4·5}$$

* Formulae were developed for estimating the difference between the two, but as the assumption seemed to be justified by the numerical results shown in tables 5 and 7, they are not reproduced here.

Making the substitutions, (2·4·1) becomes a quadratic in T_j (which form would also result from the substitution of (2·4·4) if preferred)

$$\frac{A_1 k_1 (T_j - T_s)}{l} = T_\infty^3 \left\{\frac{2Ak\sigma p\epsilon_0}{3}\right\}^{\frac{1}{2}} \left\{1 - \frac{3T_j^2}{4T_\infty^2}\right\}. \tag{2·4·6}$$

Some points arising from formula (2·4·6) are as follows. For any particular filament and lead, the only variables are T_j, T_∞, ϵ_0. The last does not vary much with temperature—in fact ϵ_0 changes only by some 7 % between 1000 and 2000 °K—so that the temperature rise of the junction $(T_j - T_s)$ varies nearly as the cube of the maximum temperature (T_∞); the formula applies almost exactly to an infinite filament with leads of equal or unequal length and closely to a relatively long filament with such leads, for example to that illustrated in figure 1 with dimensions specified in tables 2 and 5.

Another question arises in connexion with unequal leads, namely, if they result in a considerable difference in temperature between the filament ends (over 300 °C is shown in table 5, at the highest temperature), what displacement from centre would occur in the position of maximum temperature? The distribution of temperature will be symmetrical about this position, and for a long filament the displacement d from centre will be given closely by the following expression, strictly applicable to an infinite filament

$$d = \frac{1}{2T_\infty^2}\left(\frac{Ak}{\sigma p\epsilon_0}\right)^{\frac{1}{2}} \left\{\tanh^{-1}\sqrt{\frac{3z_2}{2+z_2}} - \tanh^{-1}\sqrt{\frac{3z_1}{2+z_1}}\right\}, \tag{2·4·7}$$

where $z_1 = (T_{j1}/T_\infty)^2$, $z_2 = (T_{j2}/T_\infty)^2$, T_{j1} and T_{j2} being obtained from (2·4·6) when the lengths of the leads are l_1, l_2, respectively.

The case of a short filament (of length $2L$) with unequal leads, is more complex. First, given T_m and T_s, it will usually be sufficient to calculate λ by trial from (2·1·3) using an estimated average value for T_j. Then solutions of (2·4·1) in which the two sides of this equation are given by (2·4·2) (for $l = l_1$ and $l = l_2$) and (2·4·3) yield two values T_{j1} and T_{j2}—if (2·4·3) is used the solutions would have to be graphical or by trial, while the use of (2·4·4) would admit of a simple method of solution similar to that given by (2·4·6). The value of the displacement d and a more accurate value for c could then be obtained from

$$L + d = \xi F(\phi_1, \kappa_1), \quad L - d = \xi F(\phi_2, \kappa_1), \tag{2·4·8}$$

the values ϕ_1, ϕ_2 corresponding to $z = z_1$, $z = z_2$, respectively.

2·4·2. *Peltier effect*

In this section we consider the modification of temperature distribution due to any Peltier effect between filament and leads. The heat generated or absorbed, due to this effect, is independent of the dimensions of the two metals, but depends on the magnitude and direction of the current, being equal to $\pm c\Pi_j$, where Π_j is the Peltier coefficient for temperature T_j. Instead of (2·4·1) the condition for continuity of heat flow at a junction is

$$A_1 k_1 \left(\frac{dT}{dx}\right)_j = Ak\left(\frac{dT}{dx}\right)_{jw} \pm c\Pi_j, \tag{2·4·9}$$

while in place of (2·4·6) we have for the junction temperature

$$A_1 k_1 (T_j - T_s)/l = T_\infty^3 (\tfrac{2}{3}\sigma p\epsilon_0 Ak)^{\frac{1}{2}} [1 - (3T_j^2/4T_\infty^2)] \pm c\Pi_j. \tag{2·4·10}$$

Π is a function of the temperature and can be derived from the thermoelectric force between the metals by the usual relation

$$\Pi = T\,\mathrm{d}E/\mathrm{d}T.$$

Thus if E can be expressed as a quadratic in T for the region in question, so also can Π and (2·4·10) becomes immediately soluble, the value of c being given closely enough by (2·2·1). If, however, the expression for Π involves higher powers of T, or if Π is given in graphical, instead of algebraic, form, the solution would have to be obtained by trial. We note that, with equal leads, there will be two values of T_j reversible with the current: while with unequal leads, there will be four values. For any pair of values the displacement from centre of the position of maximum temperature would be obtained on the lines indicated in the previous section.

2·5. Effect of air temperature changes on calibration

The lamps used in pyrometry are perhaps most often calibrated in terms of current, but the overall resistance or voltage is sometimes employed, and in bridge circuits some combination of current and resistance other than voltage. In this section it is proposed to consider the effect of normal variations of air temperature on these characteristics.

2·5·1. *Temperature coefficients: definitions and calculation*

Suppose that y is some specified characteristic of the lamp depending on current and/or resistance; then the temperature coefficient $(\mathrm{d}T_m/\mathrm{d}T_s)_y$ is defined as the change in T_m per unit change in T_s when y is held constant. This temperature coefficient depends on two quantities: one due to changes in radiation, which we may denote by $\partial T_m/\partial T_s$ (where both y and the junction temperatures T_j remain fixed); and the other due to changes in end conduction, denoted by $\partial T_m/\partial T_j$ (where y and T_s are fixed). To obtain the overall temperature coefficient of the lamp we then need to find the rate of change of T_j with T_s. In practice equation (2·4·6) (with $T_\infty = T_m$) is quite adequate for this purpose, and we find at once that

$$\frac{\mathrm{d}T_j}{\mathrm{d}T_s} = \frac{4T_m^2 - 3T_j^2}{4T_m^2 + 3T_j^2 - 6T_j T_s}. \qquad (2·5·1)$$

For lamps with very small gradients in the leads, in which the application of (2·4·6) would give small values for $T_j - T_s$, $\mathrm{d}T_j/\mathrm{d}T_s$ is approximately unity, and it is usually sufficient to assume that $T_j = T_s$; such conditions would apply to most lamps embodied in pyrometers. For lamps with large gradients we may use (2·5·1) and it would be necessary to determine T_j first by one of the methods outlined in § 2·4·1; this would apply to the standard strip lamps which, as will appear later, show large values for $T_j - T_s$. If the junctions of filament and leads were controlled in temperature, as with water-cooling, $\partial T_m/\partial T_s$ and $\partial T_m/\partial T_j$ give the relative effects of alterations in T_s or T_j; otherwise the overall temperature coefficient for constant y would be

$$\frac{\mathrm{d}T_m}{\mathrm{d}T_s} = \frac{\partial T_m}{\partial T_s} + \frac{\partial T_m}{\partial T_j}\frac{\mathrm{d}T_j}{\mathrm{d}T_s}. \qquad (2·5·2)$$

If the two junction temperatures T_{j1}, T_{j2} were different then there would be three terms on the right of (2·5·2); but we shall assume for simplicity of exposition that there is a common junction temperature T_j. In addition we ignore the Peltier effect, which is not significant in the present context.

We have obtained precise expressions for the required derivatives on the basis of the theory developed in §2·1. But these expressions, involving partial differentiation of an elliptic function with respect to variables occurring both in amplitude and modulus, are somewhat complex, and it is not proposed to reproduce them here, especially as numerical treatment, with finite differences, gives adequate results in the narrow range of temperature involved. The procedure may be outlined as follows. Our two basic solutions (2·1·3) and (2·1·7) $[x_j = \xi F(\phi_j, \kappa_1),\ r_j = \eta F(\theta_j, \kappa_2)]$ express the length x_j and resistance r_j from the point of maximum temperature to a junction, in terms of four main variables: thus ξ, η, κ_1, κ_2, ϕ_j, θ_j are functions of c, T_m, T_s; and ϕ_j, θ_j are also functions of T_j. We can therefore, by making small alternations in each of the four variables separately, obtain the corresponding alterations in x_j and r_j and so find approximate values for $\partial x_j/\partial c$, $\partial x_j/\partial T_m$, $\partial x_j/\partial T_s$, $\partial x_j/\partial T_j$, and for $\partial r_j/\partial c$, etc. Of course, not all of these derivatives may be required.

Constant current. In this case we simply have $x_j = \xi F(\phi_j, \kappa_1) = L$ (constant), so that

$$\frac{\partial T_m}{\partial T_j} = -\frac{\partial x_j}{\partial T_j}\bigg/\frac{\partial x_j}{\partial T_m}, \quad \frac{\partial T_m}{\partial T_s} = -\frac{\partial x_j}{\partial T_s}\bigg/\frac{\partial x_j}{\partial T_m},$$

and the temperature coefficient $(\mathrm{d}T_m/\mathrm{d}T_s)_c$ is obtained by substitution in (2·5·2).

Constant characteristic other than current. If we assume constant resistivity (ρ_1) for the leads, the resistance of a lead is $r_l = \rho_1 l/A_1$, where l is the length of the lead and A_1 its cross-sectional area; thus with equal leads the total resistance of the lamp is $2r_t$, where $r_t = r_j + r_l$. The characteristic (y, say) which we assume constant may be the resistance r_t, or some given function of c and r_t (such as the voltage $v_t = cr_t$), so that $\partial y/\partial c$, etc., may be expressed in terms of $\partial r_j/\partial c$, etc. From the simultaneous equations

$$x_j = \text{constant}, \quad y = \text{constant},$$

in which the variables are c, T_m, T_j, T_s, we have

$$\frac{\partial T_m}{\partial T_j} = -\frac{\partial(x_j, y)}{\partial(c, T_j)}\bigg/\frac{\partial(x_j, y)}{\partial(c, T_m)}, \quad \frac{\partial T_m}{\partial T_s} = -\frac{\partial(x_j, y)}{\partial(c, T_s)}\bigg/\frac{\partial(x_j, y)}{\partial(c, T_m)},$$

where $\dfrac{\partial(x, y)}{\partial(c, T)}$ is the Jacobian $\dfrac{\partial x}{\partial c}\dfrac{\partial y}{\partial T} - \dfrac{\partial x}{\partial T}\dfrac{\partial y}{\partial c}$. The temperature coefficient $(\mathrm{d}T_m/\mathrm{d}T_s)_y$ is then obtained as before by substitution in (2·5·2). It may be remarked that for most practical purposes r_l may be ignored with negligible effect on the temperature coefficient.

2·5·2. *Approximations*

For high temperatures or for long filaments the method of obtaining derivatives by taking small finite increments becomes uncertain, and also interpolation near the upper limit of the elliptic integral tables (modulus 1, amplitude 90°) becomes increasingly difficult. For such cases we have derived the following asymptotic formulae, valid for large values of χ, where

$$\chi = 2LT_m^2\left(\frac{\sigma p \epsilon_0}{Ak}\right)^{\frac{1}{2}}.$$

Write $\tau = T_j/T_m$, $a = \ln(\frac{5}{2}+\sqrt{6}) = 1{\cdot}5992\ldots$; then for constant current, resistance and voltage, respectively, we have approximately

$$\left(\frac{\partial T_m}{\partial T_j}\right)_c = 6\sqrt{6}\,\mathrm{e}^{-\chi-\tau\sqrt{6}}, \quad \left(\frac{\partial T_m}{\partial T_s}\right)_c = \left(\frac{T_s}{T_m}\right)^3; \tag{2·5·3}$$

$$\left(\frac{\partial T_m}{\partial T_j}\right)_r = -\frac{(1-\tau)\sqrt{6}}{\chi+a-2\tau\sqrt{6}}, \quad \left(\frac{\partial T_m}{\partial T_s}\right)_r = 0; \tag{2·5·4}$$

$$\left(\frac{\partial T_m}{\partial T_j}\right)_v = -\frac{(1-\tau)\sqrt{6}}{3\chi-a}, \quad \left(\frac{\partial T_m}{\partial T_s}\right)_v = \frac{2}{3}\left(\frac{T_s}{T_m}\right)^3. \tag{2·5·5}$$

These formulae improve in accuracy as χ increases. If the error in $\mathrm{d}T_m/\mathrm{d}T_s$ is not to exceed 0·05 (adequate for practical purposes), χ must be roughly greater than 4, 3·5, 4·5 for the current, resistance and voltage coefficients, respectively; for χ respectively greater than about 5, 5, 6, the error will be less than 0·01. If we let $L \to \infty$ ($\chi \to \infty$) we obtain the temperature coefficients for an infinite filament at a temperature of T_∞, namely

$$\left(\frac{\mathrm{d}T_\infty}{\mathrm{d}T_s}\right)_c = \left(\frac{T_s}{T_\infty}\right)^3, \quad \left(\frac{\mathrm{d}T_\infty}{\mathrm{d}T_s}\right)_r = 0, \quad \left(\frac{\mathrm{d}T_\infty}{\mathrm{d}T_s}\right)_v = \frac{2}{3}\left(\frac{T_s}{T_\infty}\right)^3, \tag{2·5·6}$$

formulae which are easily obtained otherwise, from (2·2·1).

The above approximations are valid at incandescent temperatures provided that χ is large enough, but it is of interest to note that for very low values of T_m the following degenerations can be derived: if $\chi_s = 2LT_s^2(\sigma p\epsilon_0/Ak)^{\frac{1}{2}}$ then when $T_m \to T_s$ we have

$$\left(\frac{\mathrm{d}T_m}{\mathrm{d}T_s}\right)_c \to 1, \quad \left(\frac{\mathrm{d}T_m}{\mathrm{d}T_s}\right)_v \to 1, \quad \left(\frac{\mathrm{d}T_m}{\mathrm{d}T_s}\right)_r \to \frac{1-\chi_s\,\mathrm{cosech}\,\chi_s}{1-\chi_s\coth\chi_s} \simeq -\tfrac{1}{2}+\tfrac{1}{40}\chi_s^2. \tag{2·5·7}$$

2·5·3. *Comparison of temperature coefficients for different lamps*

In this section we show that the temperature coefficients of a lamp are (for given temperatures) functions of $L(p/A)^{\frac{1}{2}}$, where, as usual, A and p denote the area of cross-section and the perimeter of a cross-section of the filament, of total length $2L$. We also suggest an alternative characteristic of a lamp against which the coefficients can be correlated.

First define the normalized variables

$$X' = x(p/A)^{\frac{1}{2}}, \quad C = c/(Ap)^{\frac{1}{2}}, \quad R' = r(Ap)^{\frac{1}{2}}.$$

With these variables, the general differential equation of heat flow (2·1·1) becomes

$$-\frac{\mathrm{d}}{\mathrm{d}X'}\left(k_T\frac{\mathrm{d}T}{\mathrm{d}X'}\right) = C^2\rho_T - \sigma\epsilon_T(T^4-T_s^4);$$

also, for the resistance, $A\,\mathrm{d}r = \rho_T\,\mathrm{d}x$, or $\mathrm{d}R' = \rho_T\,\mathrm{d}X'$, so that

$$-\rho_T\frac{\mathrm{d}}{\mathrm{d}R'}\left(k_T\rho_T\frac{\mathrm{d}T}{\mathrm{d}R'}\right) = C^2\rho_T - \sigma\epsilon_T(T^4-T_s^4).$$

If the boundary conditions on these equations are

$$X' = 0, \quad R' = 0 \quad \text{when} \quad T = T_m; \qquad X' = X = L(p/A)^{\frac{1}{2}}, \quad R' = R = r_j(Ap)^{\frac{1}{2}} \quad \text{when } T = T_j;$$

then the solutions of the equations (irrespective of any assumptions as to the variations of k_T, ρ_T, ϵ_T with temperature) must be of the form

$$X = f(C, T_m, T_j, T_s), \quad R = g(C, T_m, T_j, T_s),$$

the functions f and g being the same for any filament. A formal inversion and substitution then gives

$$C = f_1(X, T_m, T_j, T_s), \quad R = g_1(X, T_m, T_j, T_s). \tag{2·5·8}$$

Now let y be any specified characteristic which depends on C and/or R (we ignore the negligible resistance of the leads for this purpose, in accordance with the remark at the end of § 2·5·1); it then follows from (2·5·8) that y is expressible in the form $y = h(X, T_m, T_j, T_s)$. If now y is kept constant while T_m, T_j, T_s may vary (X is of course constant for this process) we obtain expressions for the derivatives $(\partial T_m/\partial T_s)_{T_j \text{ const.}}$ and $(\partial T_m/\partial T_j)_{T_s \text{ const.}}$, which will themselves depend on X, T_m, T_j, T_s; substitution of these expressions, together with (2·5·1), into (2·5·2), then shows that

$$(\mathrm{d}T_m/\mathrm{d}T_s)_y = h_1(X, T_m, T_j, T_s). \tag{2·5·9}$$

That is, if two filaments have the same value of $X = L(p/A)^{\frac{1}{2}}$, then at corresponding temperatures they have the same temperature coefficients.

Some simplifications ensue if, as in most pyrometer lamps, it is permissible to assume that $T_j = T_s$. For since most of the functions used above are relatively insensitive to small changes in T_s, the variables T_j and T_s can be regarded as constants, equal to normal room temperature (say 15 °C); (2·5·8) and (2·5·9) then take the form

$$C = c/(Ap)^{\frac{1}{2}} = f_2(X, T_m), \tag{2·5·10}$$

$$R = r_j(Ap)^{\frac{1}{2}} = g_2(X, T_m), \tag{2·5·11}$$

$$(\mathrm{d}T_m/\mathrm{d}T_s)_y = h_2(X, T_m). \tag{2·5·12}$$

Equations (2·5·10) and (2·5·11) enable us to give standardized current and resistance calibrations which can be applied to any pyrometer lamp with a given value of X; moreover, a further corollary, obtained by multiplying these equations together, is

$$V = CR = cr_j = v_j = f_3(X, T_m); \tag{2·5·13}$$

that is, two pyrometer lamps with the same value of X have the same voltage calibration.

Finally, we examine the possibility of replacing X in the above formulae by some easily measured characteristic of the pyrometer lamp. For this purpose we suggest the ratio of the currents required to match a black body held at two fixed temperatures T_1 and T_2 respectively, for instance the melting points of gold and palladium, which are two fixed points on the International Temperature Scale. Thus, with T_s given, let the corresponding values of c be c_1 and c_2, where

$$c_2/c_1 = C_2/C_1 = q.$$

Using (2·5·10), since we suppose T_s fixed and $T_j = T_s$, we then have

$$q = f_2(X, T_2)/f_2(X, T_1);$$

since T_1, T_2 are fixed numbers, this means that there is a direct functional relation between X and q, so that the formulae (2·5·10) to (2·5·13) each have an alternative form with q in place of X. For applications of these results to pyrometer lamps see § 3·2·1.

2·6. Thomson effect

The physical constants of tungsten which have so far been considered in relation to the temperature distribution in a filament, have been based on a considerable amount of experimental evidence. The Thomson effect, however, has been studied only by Worthing (1915), who had to rely on the optical pyrometer for measuring the small differences in temperature on which determination of the effect depended—such measurements can now be made with much greater ease and precision by means of photocell apparatus. In this section the effect, on distribution, of assumed variations of the Thomson coefficient with temperature, is briefly considered with a view to assisting further experimental work.

The Thomson effect in an element of filament does not depend on its length ($\mathrm{d}x$) but only on the temperature difference ($\mathrm{d}T$) between its ends. If ψ_T is the Thomson coefficient at a temperature T, the energy absorbed or liberated (according to the direction of the current) is $c\psi_T\,\mathrm{d}T$. Thus the basic energy equation is

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak_T\frac{\mathrm{d}T}{\mathrm{d}x}\right)=\frac{c^2\rho_T}{A}-\sigma p\epsilon_T(T^4-T_s^4)\pm c\psi_T\frac{\mathrm{d}T}{\mathrm{d}x}. \qquad (2{\cdot}6{\cdot}1)$$

The last term of this equation is of small effect, when compared with the others, and can be neglected for a first approximation. The value of $\mathrm{d}T/\mathrm{d}x$ obtained on integration can then be 'fed back' into the last term, precisely as explained in §2·3, and so give a second approximation, bringing in the Thomson coefficient. The process can be repeated if thought necessary.

Thus for the first approximation we have from §2·1·1

$$\mathrm{d}T/\mathrm{d}x = N(T^6-3BT^2+M)^{\frac{1}{2}},$$

where $B = T_s^4+c^2\rho_0/A\sigma p\epsilon_0, \quad N=(\sigma p\epsilon_0/3Ak)^{\frac{1}{2}}, \quad M = 3BT_m^2-T_m^6,$

T_m being the maximum temperature in the filament. Substituting this value of $\mathrm{d}T/\mathrm{d}x$ in the last term of (2·6·1) and integrating between T and T_m, we obtain

$$(\mathrm{d}T/\mathrm{d}x)^2 = N^2(T^6-3BT^2+M)\pm(2cN/Ak)\,G(T),$$

where
$$G(T)=\int_T^{T_m}\psi_T(T^6-3BT^2+M)^{\frac{1}{2}}\,\mathrm{d}T. \qquad (2{\cdot}6{\cdot}2)$$

Inverting and expanding as far as the second term, we have

$$\frac{\mathrm{d}x}{\mathrm{d}T}\simeq\frac{1}{N}(T^6-3BT^2+M)^{-\frac{1}{2}}\pm\frac{c}{AkN^2}G(T)\,(T^6-3BT^2+M)^{-\frac{3}{2}}.$$

For any given temperature T there will be two values of x (x_1 and x_2) corresponding to the two current directions and, on integrating the last equation, the difference between them, taking an origin at the cold end* of the filament at a temperature T_0, would be

$$x_2-x_1=\frac{6c}{\sigma p\epsilon_0}\int_{T_0}^{T}G(T)\,(T^6-3BT^2+M)^{-\frac{3}{2}}\,\mathrm{d}T. \qquad (2{\cdot}6{\cdot}3)$$

In particular this would give, for $T=T_m$, the shift of the maximum on reversal of the current, due to the Thomson effect.

* Here we ignore the Peltier effect, though this can be corrected for, or else eliminated by suitable choice of lead material (see §3·1·7).

To apply this result we should need to assume some form for the relation of Thomson coefficient to temperature, say a polynomial

$$\psi_T = \alpha + \beta T + \gamma T^2 + \ldots, \tag{2·6·4}$$

and in general (2·6·3) would call for a double numerical integration, though $G(T)$ would in fact be expressible in terms of elliptic functions. In the case of a filament sufficiently long to be regarded, without serious error, as infinite, considerable simplification ensues, in that we can put $T_m = T_\infty$ and so

$$T^6 - 3BT^2 + M = (T_\infty^2 - T^2)^2 (2T_\infty^2 + T^2).$$

$G(T)$ is then expressible in terms of hyperbolic functions for any form (2·6·4) of ψ_T and the evaluation of $x_2 - x_1$ requires a single numerical integration.

2·7. Approach to the steady state

The approach to the steady state of temperature distribution, of lamps used in pyrometry, is of importance in that considerable error may result from readings taken before equilibrium is reached. This applies particularly to standard strip lamps for which as much as 30 min may have to be allowed, from switching on, for safety in this respect.

The treatment of such a problem is quite impracticable without simplifying assumptions, and we here assume the filament to be infinitely long and to be run either at constant current or at constant voltage per unit length. With a finite filament these two conditions would be fulfilled by a source of power, of constant e.m.f. and zero internal resistance, connected respectively to a lamp with a very high series resistance, or directly to the lamp itself. In practice, a pyrometer circuit usually consists of a high-capacity accumulator with a resistance in series, and so would fall between the extremes of the two conditions, but would normally be closer to that of constant current, since the series resistance would in general be considerably greater than that of the lamp. In the case of a finite filament the temperature at its ends has also to be taken into account. If this remains sensibly constant, as with very heavy or water-cooled leads, the response, from switching on, might be expected to follow closely that of an infinite filament of the same cross-section—an observation quoted in § 3·1·6 below tends to confirm this view. On this supposition, the data for an infinite filament would provide upper and lower limits for the minimum response time for a finite filament and serve to indicate the degree of avoidable creep resulting from any gradual temperature change of the filament ends, however caused. We deal below with the response of an infinite filament under the two conditions named.

2·7·1. *Response of infinite filament: constant current*

In this section we make the same assumptions as before, namely, $\rho_T = \rho_0 T$, $\epsilon_T = \epsilon_0 T$, and we further take as constant the heat capacity per unit volume (h). Then the following equation expresses the fact that the heat generated by a constant current c_∞ in time dt less that lost by radiation equals the heat required to raise the temperature of the filament by dT:

$$[c_\infty^2 \rho_T / A - \sigma p \epsilon_T (T^4 - T_s^4)]\, dt = Ah\, dT. \tag{2·7·1}$$

Since c_∞ is the current required to maintain the filament at a temperature T_∞ we have, from (2·2·1),

$$c_\infty^2 \rho_0 = A\sigma p\epsilon_0 (T_\infty^4 - T_s^4);$$

if we eliminate c_∞ from (2·7·1) and write

$$G = Ah/4\sigma p\epsilon_0 T_\infty^4, \quad \tau = T/T_\infty,$$

equation (2·7·1) takes the form

$$4G\, d\tau/dt = \tau(1-\tau^4)$$

of which the solution is

$$t = G \ln(\tau^{-4}-1) + \text{constant}.$$

For the relation between t and T when the current is switched on to the filament at the temperature of the surroundings (T_s) we then have

$$t = G \ln\{(\tau^{-4}-1)/(\tau_s^{-4}-1)\}. \tag{2·7·2}$$

This relation has been used for calculating one of the response curves shown in figure 3, p. 491. However, the main concern in practice is with the time taken for the filament to reach, from cold, to within a small fixed value Δ (say 1 or 2 °C) of the final temperature T_∞. For this we have derived from (2·7·2) the following formula (expressed in the original symbols) which is convenient for calculation and gives a very close approximation:

$$t_{c(\Delta)} = (Ah/4\sigma p\epsilon_0 T_\infty^4) \ln(T_\infty^5/4T_s^4\Delta). \tag{2·7·3}$$

This form has been used to obtain a curve on each of figures 4 and 5.

It is of interest to consider the bearing of two factors, namely, dimensions of filament and value of T_∞, on the times given by (2·7·2) and (2·7·3). We note that in the two formulae the time is directly proportional to A/p (cross-section ÷ perimeter). This would allow the time of response for any filament to be deduced from the corresponding time for any other filament, run at the same temperature; and also an interesting comparison to be made between cylindrical and flat filaments. Thus for a cylindrical filament of diameter b, $A/p = \frac{1}{4}b$, while for a flat filament of width w and thickness u, $A/p = \frac{1}{2}u(1+u/w) = \frac{1}{2}u$ approximately, if u/w is small. For certain purposes the size of image presented by the filament is of importance in optical pyrometry. If we sight normally on the wider side of the strip it will present an identical aspect to that of a cylindrical filament if $w = b$, but the strip would obviously have a much quicker response.

With regard to the effect of differing maximum temperatures, we observe that the logarithmic term of T_∞ changes but slowly when compared with the term T_∞^4, and hence the times given by (2·7·3) may be taken to vary roughly as the inverse fourth power of T_∞.

2·7·2. *Response of infinite filament: constant voltage per unit length*

If v_∞ is the constant voltage per unit length corresponding with the final temperature T_∞, the basic equation will be

$$[v_\infty^2 A/\rho_T - \sigma p\epsilon_T(T^4 - T_s^4)]\, dt = Ah\, dT,$$

where v_∞ is given by

$$v_\infty^2 = (c_\infty \rho_0 T_\infty/A)^2 = \sigma p\epsilon_0 \rho_0 T_\infty^2 (T_\infty^4 - T_s^4)/A.$$

Eliminating v_∞, we have

$$\sigma p\epsilon_0 [T_\infty^2 (T_\infty^4 - T_s^4) - T^2(T^4 - T_s^4)]\, dt = AhT\, dT.$$

If we change the variable to $\tau = T/T_\infty$ and write

$$G = Ah/4\sigma p\epsilon_0 T_\infty^4, \quad \tau_s = T_s/T_\infty,$$

the equation takes the form

$$4G\tau\,\mathrm{d}\tau/\mathrm{d}t = (1-\tau^2)(\tau^4+\tau^2+1-\tau_s^4).$$

On integration this yields the following, subject to the condition $\tau_s^4 < \frac{3}{4}$ which always applies in practice

$$t = f(\tau) + \text{constant}, \tag{2·7·4}$$

where
$$f(\tau) = \frac{G}{3-\tau_s^4}\left\{\ln\frac{\tau^4+\tau^2+1-\tau_s^4}{\tau^4-2\tau^2+1} + \frac{6}{\surd(3-4\tau_s^4)}\tan^{-1}\frac{2\tau^2+1}{\surd(3-4\tau_s^4)}\right\};$$

then the time t taken for the filament to heat up from temperature $T_0(\tau_0 = T_0/T_\infty)$ to temperature T is

$$t = f(\tau) - f(\tau_0). \tag{2·7·5}$$

This relation has been used for calculating one of the curves in figure 3. However, our main concern, as mentioned above, is with the time taken from cold (T_s) for the filament to get within a small amount Δ of T_∞. For this we have derived the following approximate formula, which permits of easy calculation

$$t_{v(\Delta)} = \frac{Ah}{12\sigma p\epsilon_0 T_\infty^4}\left\{\ln\left(\frac{3T_\infty^2}{4\Delta^2}\right) + \frac{\pi}{\surd 3} - 6\left(\frac{T_s}{T_\infty}\right)^2\right\}. \tag{2·7·6}$$

This form has been used to obtain curves shown in figures 4 and 5. Similar remarks to those of the preceding section, as to the effect on response of filament dimensions and value of T_∞, would apply here; but the rule of variation of t with T_∞^{-4} would not yield so close a result, and the effect of variations of T_s would be somewhat greater.

3. VERIFICATIONS AND APPLICATIONS OF THE THEORY

The preceding section gives the full development of our theory built on three basic assumptions and includes a number of formulae suitable for calculation. These expressions are quite general and could be used to calculate the characteristics of many types of lamp, but in this section they are applied only to the two main types used in pyrometry, in which we are primarily interested, namely: (i) strip filament lamps used as standards for reproducing the International Temperature Scale (Comm. Int. des Poids et Mésures 1948); (ii) lamps with finer filaments, either strip or wire, embodied in optical pyrometers. Most of the calculations relate to a lamp of type (i) above since, as indicated in §2·4·1, this serves the better to illustrate the special problems of leads, etc. The scheme of the calculations has been planned with the following objectives in view: to establish the validity of our basic assumptions; to allow comparison with an earlier theory (that of Ribaud & Nikitine—see appendix); to supply data and suggestions of value to designers and users of pyrometric lamps.

The first objective, which is obviously the most important, calls for special comment. Of possible methods of verification, the most convincing is that of comparison of calculation with the findings of experiment. We have applied this test wherever possible and, as will be seen, it has tended to justify our three assumptions. But the experimental evidence is somewhat limited and consequently we have sought elsewhere for justification, particularly of our two more important simplifications, namely, those for resistivity and emissivity. The recent developments in computer technique, giving direct numerical solutions of differential equations free from any simplifications, seemed ideal for the purpose. Accordingly, a series of calculations, designed to throw light on various aspects of the problem, were undertaken at our request by the National Physical Laboratory. As will be seen, a comparison of these with our own calculations has served to validate the two assumptions. It has also been of value in indicating the closeness with which our simpler formulae (for infinitely long filaments) might be expected to apply to the relatively long filaments often used in pyrometry.

With the numerical method playing so important a role, the question may be asked whether it could not entirely replace our algebraic treatment. On this point it may be remarked that the latter has dealt with some nine independent variables (in addition to the physical constants of tungsten) and that general expressions, in terms of some of these, have been derived for such entities as the temperature distribution in filament and leads; the total resistance of the filament; its temperature coefficients of calibration; its speed of response. These expressions, with the relatively unimportant exception of (2·6·3), all permit of ready computation and, since they apply to any type of tungsten vacuum-lamp, cover large ranges of the variables—in pyrometric lamps alone, ranges in excess of 10 to 1 occur in filament dimensions and current. With the numerical method it would obviously be impracticable to provide tabular material to cover all possible, or even likely, permutations of the several variables. But though the numerical method cannot be a substitute for the full algebraical treatment, it can no doubt be of great value in problems of limited scope such as, for example, that indicated in the last paragraph of § 3·1·5 below.

For the purpose of our calculations the main properties of tungsten are taken to be as follows:

Thermal conductivity. A constant value of 1·14 W cm/°C. This is Langmuir's value from table 1; owing to the spread of results there shown, the effect of departure from the chosen value is very briefly considered.

Emissivity or absorptivity (ϵ) *and resistivity* (ρ). The values from table 67 of Ribaud (1931) based on the work of Forsythe & Worthing (1925) and of Zwikker (1925).

The black-body constant of total radiation is taken as

$$\sigma = 5{\cdot}74 \times 10^{-12}\,\mathrm{W\,cm^{-2}\,{}^\circ C^{-4}}.$$

Other data, such as dimensions of filament and leads, and thermoelectric coefficients, are given in the appropriate sections. As explained in § 1·2·2, the true temperature (T) at a point on the filament is expressed in absolute measure (degrees Kelvin); where apparent or brightness temperatures are given, these, following usual practice, are stated consistently in degrees centigrade.

3·1. Standard strip lamps

Tungsten filaments in the form of flat strip are now universally preferred to those of circular section, for pyrometric standards. The reasons for this preference have been well set out by Ribaud (1931) who has given general data, very useful for designers of such lamps, in his tables 47 and 48. Our object is to examine in greater detail the characteristics of a single type of lamp, namely, that illustrated in figure 1. This type is based on a design by the NELA Research Laboratory, U.S.A. (Forsythe 1923), and is commonly used in this country; we believe that it does not differ greatly from that adopted elsewhere. The dimensions of the filament are given in the heading of table 2.

For this lamp we treat the problem of temperature distribution and allied matters in increasing stages of complexity. Thus we first take the temperature (T_j) of the filament ends to be fixed and equal (which state can be attained—see §3·1·7) and ignore the factors of unequal leads and of Peltier and Thomson effects, which are subsequently considered in successive stages. Temperature coefficients of calibration and the approach to the steady state are also dealt with and, finally, possible alterations in design are considered.

TABLE 2. CALCULATIONS OF CURRENT AND RESISTANCE OF STRIP LAMP FILAMENT

Dimensions of filament: length 6·2 cm, section 0·131 × 0·0064 cm.
Conditions: T_j = 400 °K, T_s = 288 °K.
Calculations:
N.P.L., by National Physical Laboratory without assumptions for ρ and ϵ.
R. & S., based on present paper (first approximation, §2·1).
R. & S. (2), based on present paper (second approximation, §2·3).
R. & N., based on theory of Ribaud & Nikitine (see appendix).
∞, current for infinite filament of cross-section as above.

characteristic	method	maximum true temperature °K (apparent temperature °C) 500 —	800 —	1100 (779 °C)	1400 (1050 °C)	1569 (1200 °C)
current	∞	0·60	1·55	2·92	4·80	6·06
c (A)	N.P.L.	1·39	2·40	3·36	4·91	—
	R. & S.	1·38	2·37	3·33	4·90	6·09
	R. & S. (2)	—	—	3·34	4·91	—
	R. & N.	1·02	1·95	3·14	4·85	6·06
resistance	N.P.L.	0·073	0·115	0·165	0·229	—
$2r_j$ (Ω)	R. & S.	0·074	0·120	0·171	0·233	0·279
	R. & N.	0·072	0·119	0·176	0·243	0·285

3·1·1. *Characteristics of strip lamp: fixed junction temperatures*

The temperature distribution along the filament is taken to be symmetric about its centre and the temperatures (T_j) at its ends to be equal and constant. We then calculate, for any particular value of the maximum temperature (T_m), the current required, the resistance of the filament, and its temperature distribution, on the basis of the assumptions $\rho_T = \rho_0 T$, $\epsilon_T = \epsilon_0 T$, k_T constant. For this purpose ρ_0 and ϵ_0 are not taken as absolute constants, but to be of such values as to make the resistivity and emissivity agree with the accepted figures for the particular T_m. This is done in recognition of the overriding importance of the conditions near the centre of the filament.

The first step in calculation is to determine the current required to fit the boundary conditions, as explained in §2·1·1. The remainder of the calculations do not call for special comment. In addition, corresponding figures are given based on the theory of Ribaud & Nikitine (1927), slightly modified as explained in the appendix to this paper. Both sets of calculations have involved simplifying assumptions and as a check on these, computations were made by the Mathematics Division of the National Physical Laboratory from the original equation (2·1·1), taking our constant value for k, but published data (Ribaud 1931, table 67) for the variations of resistivity and emissivity with temperature.

The results of the several calculations of current and resistance are given in table 2 for five values of T_m. Only the last three values (expressed as true temperatures °K) are above the visible limit and for these the figures given in brackets (°C) are the corresponding apparent temperatures obtained by applying corrections for emissivity and an 8 % absorption for the single thickness of glass through which the filament is viewed.

With regard to current, it will be seen that the N.P.L. and R. & S. values are in satisfactory agreement and show the approach, with rising temperatures, to the values for an infinite filament given in the first line of the table. The R. & N. values show considerable departures from the others, especially at the lower temperatures. With regard to resistance, neither the R. & S. nor R. & N. figures differ greatly from those of N.P.L.

It is also of interest to compare calculated values with those obtained by experiment. In practice the value of T_j is not fixed as in table 2, but rises with T_m, and moreover is different for the two ends of a filament with unequal leads. Referring to a later table 5 we see that the mean value of T_j, for $T_m = 1400$ °K, is of the order of 420 °K, while that for $T_m = 1100$ °K can similarly be shown to be about 360 °K. In table 3 the experimental values of current and resistance for a filament of the dimensions previously given are compared with N.P.L. calculated values applicable to T_j temperatures of 350 and 450 °K.

TABLE 3. COMPARISON OF EXPERIMENTAL AND CALCULATED VALUES

Dimensions of filament as in table 2.
expt. are experimental values.
N.P.L. calculations with $T_s = 288$ °K.

	current c (A)			resistance $2r_j$ (Ω)		
	calc. for T_j			calc. for T_j		
T_m (°K)	350 °K	450 °K	expt.	350 °K	450 °K	expt.*
1100	3·40	3·32	3·37	0·161	0·169	0·167
1400	4·92	4·90	4·90	0·225	0·232	0·230

* For a strict comparison, the total resistance of the leads (about 0·002 Ω) should be subtracted from the figures in this column.

The fact that the experimental values fall between the extremes of those calculated may be taken as endorsing the method of treatment and the numerical values assumed for the constants. For instance, it can easily be shown that the effect of a 10 % difference in the initial choice of the thermal conductivity k would be to change c (in the same direction) by an amount varying from 2·0 % at 1100 °K to 0·54 % at 1400 °K, and to change r (in the opposite direction) by 0·4 % at 1100 °K, up to 0·7 % at 1400 °K.

The distribution of temperature along the filament, as given by the several methods of calculation, is also considered. For this purpose fixed temperatures have been taken for T_m (1400 °K) and T_j (400 °K), T_s being 288 °K, and the N.P.L. calculations, shown in the first line of table 4, have been treated as standard. The R. & S. (ell.) figures are based on the elliptic formula (2·1·5) in this paper, and the R. & S. (hyp.) figures on the hyperbolic formula (2·2·2), for the latter a value of T_∞ being chosen so as to give the temperature of 1400 °K at the distance of 3·1 cm from the cold end (i.e. at the centre of the filament). In addition to this full comparison we also give the figures for R. & S. (ell.) and R. & S. (hyp.) at a lower value (1013 °K) for T_m. It may be added that the T_∞ for 1400 °K was 1407·7 °K, and for 1013 °K it was 1067·2 °K.

It will be seen that the R. & N. values at 1400 °K (1050 °C apparent) depart considerably from the standard N.P.L. values, while the two sets of R. & S. values are much closer to standard with little to choose between them. The divergencies between the two R. & S. values naturally become larger towards the bottom of the scale—see figures for 1013 °K (700 °C apparent)—but are not such as to invalidate the use of the simpler hyperbolic formula to get a fair idea of distribution throughout the range of the lamp.

Table 4. Calculations of temperature distribution along strip filament

Dimensions of filament as in table 2.
Conditions: T_j = 400 °K, T_s = 288°K.
Calculations:
N.P.L., by National Physical Laboratory without assumptions for ρ and ϵ.
R. & S. (ell.), based on present paper with elliptic formulae.
R. & S. (hyp.), based on present paper with hyperbolic formulae.
R. & N., based on theory of Ribaud & Nikitine.

method	temperatures (°K) at distance (cm) from cold end							
	3·1	2·6	2·1	1·6	1·1	0·6	0·1	0
N.P.L.	1400	1393	1368	1304	1163	897	495	400
R. & S. (ell.)	1400	1393	1372	1307	1170	907	497	400
R. & S. (hyp.)	1400	1389	1362	1299	1162	905	495	400
R. & N.	1400	1397	1384	1352	1271	1069	568	400
R. & S. (ell.)	1013	1001	963	893	785	634	443	400
R. & S. (hyp.)	1013	980	930	856	751	611	442	400

Table 5. Junction temperatures and displacement (d) of maximum

Dimensions of filament as before, *of nickel leads* 0·29 cm diameter, 3·5 and 8·5 cm long.
Calculations for T_s = 288 °K:
N.P.L., by National Physical Laboratory without assumptions for ρ and ϵ.
R. & S. (hyp.), based on present paper with hyperbolic formulae.

method	maximum temperature		junction temperatures (°K)			d (mm)
	true (°K)	apparent (°C)	3·5 cm	8·5 cm	difference	
N.P.L.	1400	1050	367	471	104	0·564
R. & S. (hyp.)	1400	1050	368	476	108	0·569
R. & S. (hyp.)	1013	700	316	356	40	0·580
	1569	1200	403	554	151	0·561
	2150	1700	581	935	354	0·528

3·1·2. *Junction temperatures: unequal leads*

The leads of the strip lamp (see simplified representation in figure 2) are assumed to be of pure nickel, 0·29 cm in diameter and respectively 3·5 and 8·5 cm long. For given values of T_m and T_s calculations (designated by R. & S. (hyp.)) are made of the two junction temperatures (from (2·4·6)) and the displacement (d) from centre of the maximum temperature (from (2·4·7)). In these formulae we take $T_\infty = T_m$, and any disturbance due to Peltier or Thomson effects is at present ignored. The main example is for a T_m of 1400 °K, where comparison is made with the standard N.P.L. values. The other temperatures cover the normal range of the lamp (700 to 1700 °C apparent).

It will be seen that at 1400 °K there is satisfactory agreement between our figures and the standard N.P.L. ones. This has encouraged us to apply the hyperbolic formula over the whole range of the lamp. For the highest temperature (2150 °K), with the approximate formula pushed to an extreme limit and the uncertainty (already mentioned) as to the effective lengths of the leads, the figures must be taken as giving only rough estimates; nevertheless, they are consistent with the experimental observation that the filament ceases to give any visible radiation at about 1 mm and within $\frac{1}{4}$ mm from the respective ends of the filament, the corresponding figures calculated from the junction temperatures, and taking the lowest visible temperature as 650 °C apparent (959 °K true), being 1·1 and 0·1 mm.

3·1·3. *Peltier effect*

In this section we carry the analysis a stage beyond § 3·1·2 by considering the modifications due to the Peltier effect. As pointed out in § 2·4·2, the Peltier coefficient (Π) can be derived from the thermoelectric relations of the two metals; thus, taking the published data* for the thermoelectric force (E) between tungsten and nickel, we have derived table 6 of

Table 6. Peltier coefficient between tungsten and nickel

T (°K)	300	400	500	600	700	800	900	1000	1100	1200
Π (mV)	6·9	12·4	16·6	18·9	22·9	29·6	38·5	48·3	58·4	69·2

Table 7. Effect of current reversal on junction temperatures and displacement of maximum

Dimensions of filament and leads as in tables 2 and 5.
Calculations for $T_s = 288$ °K:
N.P.L., by National Physical Laboratory without assumptions for ρ and ϵ.
R. & S. (hyp.), based on present paper with hyperbolic formulae.

current direction	method	T_m (°K)	T_{j_1} (°K)	T_{j_2} (°K)	$T_{j_2}-T_{j_1}$	d (mm)
$j_1 \rightarrow j_2$	N.P.L.	1400	363	486	123	0·669
	R. & S. (hyp.)	1400	364	491	127	0·674
$j_2 \rightarrow j_1$	N.P.L.	1400	371	457	86	0·463
	R. & S. (hyp.)	1400	373	461	88	0·468
$j_1 \rightarrow j_2$	R. & S. (hyp.)	1013	314	361	47	0·68
		1569	397	576	179	0·67
		2150	564	1031	467	0·71
$j_2 \rightarrow j_1$	R. & S. (hyp.)	1013	318	351	33	0·48
		1569	410	534	124	0·46
		2150	598	867	269	0·40

values of $\Pi = T\,dE/dT$. These figures are liable to be affected considerably by the purity of the metals, but the precise values are not of too much importance here, since our main concern is a comparison of the results yielded by our formulae with the standard N.P.L. figures obtained, as before, without our assumptions as to ρ and ϵ. The equations used are (2·4·10) and (2·4·7), with $T_\infty = T_m$, the solution of (2·4·10) being obtained by trial by means of table 6; for the results at 1400 °K a law of the form $\Pi = aT+bT^2+cT^3$ was taken, matched over the range 300 to 600° K. Specimen results are given in table 7. The degree of agreement between the two sets of values for $T_m = 1400$ °K indicates that our approximate treatment is satisfactory.

It is of interest to use the table to isolate the effects of the two factors—unequal leads and Peltier effect. Thus the effect of current reversal on the N.P.L. figures for 1400 °K may be written

$$T_{j_2}-T_{j_1} = 104{\cdot}5 \pm 18{\cdot}5\ \text{°K}, \quad d = 0{\cdot}566 \pm 0{\cdot}103\ \text{mm};$$

* *Temperature*, Symposium of Amer. Inst. of Physics (1941).

since the N.P.L. calculation without Peltier effect (see table 5) makes $T_{j_2}-T_{j_1} = 104\,°K$ and $d = 0{\cdot}564$ mm, we may conclude that the Peltier disturbance at the junctions in this case is of the order of $\pm 19\,°C$ and near the centre the negligible amount of $\pm 0{\cdot}1$ mm. The other figures do not call for special comment except that the displacements of T_m from centre all fall within narrow limits, the extreme variation being only $\pm 0{\cdot}16$ mm.

3·1·4. *Temperature coefficients*

In table 8 we give some results of applying the theory of §2·5 to obtain temperature coefficients for the standard strip lamp. The approximations of §2·5·2 can be applied for $T_m > 1050\,°C$ apparent, and, indeed, may be considered adequate in practice for $T_m > 800\,°C$ apparent. All the calculations are for equal leads of length l, but the effect of different choices of l is indicated for the extreme temperatures; in both cases it will be seen that the variation in l of $\pm 2{\cdot}5$ cm causes variations in the temperature coefficients of at most 0·02, and usually considerably less; moreover, the values of the temperature coefficients for $l = 6{\cdot}0$ cm is the mean of the values for $l = 3{\cdot}5$ cm and $l = 8{\cdot}5$ cm. We infer that if we have unequal leads in this type of lamp, then calculations made for equal leads of mean length will yield temperature coefficients indistinguishable from those obtained by more complex calculations with unequal leads; a specimen calculation (not shown) confirms this.

Table 8. Temperature coefficients of standard strip lamp

Dimensions of filament as in table 2; equal nickel leads 0·29 cm diameter, length l.
Calculations for $T_s = 288\,°K$:
(*a*) based on elliptic formulae (2·1·3), (2·1·7), as described in §2·5·1;
(*b*) from approximations in §2·5·2, where $\chi = 2LT_m^2(\sigma p\epsilon_0/Ak)^{\frac{1}{2}}$.
$(\mathrm{d}T_m/\mathrm{d}T_s)_c$ gives rise in T_m for 1 °C rise in T_s, with filament run at constant current; similarly $(\mathrm{d}T_m/\mathrm{d}T_s)_r$ for constant resistance and $(\mathrm{d}T_m/\mathrm{d}T_s)_v$ for constant voltage across the filament.
T_m in °C apparent, with true temperatures °K in ( ).

T_m	l (cm)	T_j (°K)	c (A)	χ	$(\mathrm{d}T_m/\mathrm{d}T_s)_c$ (*a*)	$(\mathrm{d}T_m/\mathrm{d}T_s)_c$ (*b*)	$(\mathrm{d}T_m/\mathrm{d}T_s)_r$ (*a*)	$(\mathrm{d}T_m/\mathrm{d}T_s)_r$ (*b*)	$(\mathrm{d}T_m/\mathrm{d}T_s)_v$ (*a*)	$(\mathrm{d}T_m/\mathrm{d}T_s)_v$ (*b*)
700 (1013)	6·0	336	3·076	—	+0·33	—	−0·46	—	$-0{\cdot}03_4$	—
	3·5	316	3·055	—	+0·34	—	−0·47	—	$-0{\cdot}03_3$	—
	8·5	356	3·097	—	+0·32	—	−0·45	—	$-0{\cdot}03_5$	—
800 (1122)	6·0	354	3·459	3·428	+0·19	+0·23	−0·42	−0·47	$-0{\cdot}07_3$	−0·18
1050 (1400)	6·0	423	4·900	5·518	$+0{\cdot}03_5$	$+0{\cdot}03_5$	−0·29	−0·29	$-0{\cdot}09_4$	$-0{\cdot}10_3$
1200 (1569)	6·0	481	—	7·044	—	$+0{\cdot}01_2$	—	−0·22	—	$-0{\cdot}07_8$
1700 (2150)	6·0	768	—	13·16	—	$+0{\cdot}00_2$	—	−0·11	—	$-0{\cdot}03_5$
	3·5	581	—	13·16	—	$+0{\cdot}00_2$	—	−0·13	—	$-0{\cdot}04_3$
	8·5	935	—	13·16	—	$+0{\cdot}00_2$	—	−0·09	—	$-0{\cdot}02_8$

The values of $(\mathrm{d}T_m/\mathrm{d}T_s)_c$ between $T_m = 700\,°C$ and $1050\,°C$ apparent may be compared with the observational values given in Fig. 4(*a*) of Barber (1946*a*); the dimensions of his lamp are only slightly different from ours and his observed temperature coefficients of current range from 0·36 at 700 °C to about 0·01 at 1050 °C, so that they follow closely the corresponding values in table 8.

We also remark that an examination of the components of the temperature coefficients has shown that at incandescent temperatures the 'radiation temperature coefficient' $\partial T_m/\partial T_s$ is uniformly small, the maximum value being $0{\cdot}02_6$, for $(\partial T_m/\partial T_s)_c$ at 700 °C. Thus whenever a temperature coefficient is large enough for practical considerations, most of its

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effect comes from changes in junction temperatures, indicating that if these temperatures can be accurately controlled the effect on lamp calibrations of reasonable changes in air temperature will be negligible.

3·1·5. *Thomson effect*

In §2·6 consideration was given to the effect, on temperature distribution, of assumed variations of the Thomson coefficient (ψ_T) with temperature. In this section we take two such relations, arbitrarily chosen, and apply them to calculate the values of the Thomson coefficient corresponding with several displacements from centre of the maximum temperature T_m in the filament, assuming equal leads; these displacements are of the order indicated by the experiments of Barber (1946*a*). The calculations are made for fixed values of T_m, T_j and T_s, and in the assumed absence of the Peltier effect. The results are obtained from equation (2·6·3) by double numerical integration, and are compared in table 9 with standard N.P.L. calculations free from our usual simplifications for resistivity and emissivity.

TABLE 9. CALCULATION OF THOMSON COEFFICIENT FROM DISPLACEMENT OF T_m

Dimensions of filament as in table 2.
Conditions T_m = 1400 °K, T_j = 400°K, T_s = 288 °K.
Assumed relation of ψ_T *with* T:
(*a*) $\psi_T = \psi_0$; (*b*) $\psi_T = \psi_m T/T_m$ (ψ_0, ψ_m constants).
Calculations:
N.P.L., by National Physical Laboratory without assumptions for ρ and ϵ.
R. & S., based on present paper (§2·6).

displacement of T_m (mm)	$10^5\psi_0$ (V/°C) relation (*a*)		$10^5\psi_m$ (V/°C) relation (*b*)	
	N.P.L.	R. & S.	N.P.L.	R. & S.
−0·3	−0·62	−0·64	−0·74	−0·77
−0·4	−0·84	−0·85	−0·98	−1·02
−0·5	−1·05	−1·07	−1·23	−1·28

The agreement between the two sets of values is satisfactory, but they apply only to a known shift of T_m and the two assumptions for ψ_T. Actually, as pointed out in §2·6, very little is known of the Thomson effect in tungsten. There would therefore seem to be a demand for fresh experiment which might conceivably be on the following lines; with a strip lamp of the type of figure 1, but with leads having no Peltier effect against tungsten, and the junction temperatures (T_j) kept equal and constant, the shift in temperature on reversal of the current would be determined by photocell at a number of points along the filament, the process being repeated for several steady values of the maximum temperature (T_m). The problem would then be to find the formula for the Thomson coefficient best fitted to the mass of observations. With ordinary arithmetic processes based on our theory, this task would be somewhat laborious, but it could no doubt be handled easily with a modern computer, the only variables being the formula under trial and the value of T_m.

3·1·6. *Speed of response of strip lamp*

In this section the results of §2·7 are applied to the case of a filament of infinite length and of the cross-section taken above, namely 0·131 × 0·0064 cm. Figure 3 shows the forms of curve (*c* and *v*) for heating up the filament from cold (15 °C) to an apparent temperature of 1200 °C (1569 °K true), when run at constant current and constant voltage per unit length, respectively, calculated from (2·7·2) and (2·7·5), respectively. An observation, *o*, is also shown on figure 3 for an actual lamp of the dimensions given in tables 2 and 5. This lamp was run so as to reach the same final temperature (1200 °C), being connected to a 6 V battery through a fixed resistance of about 1 Ω. The final resistance of the lamp was 0·28 Ω

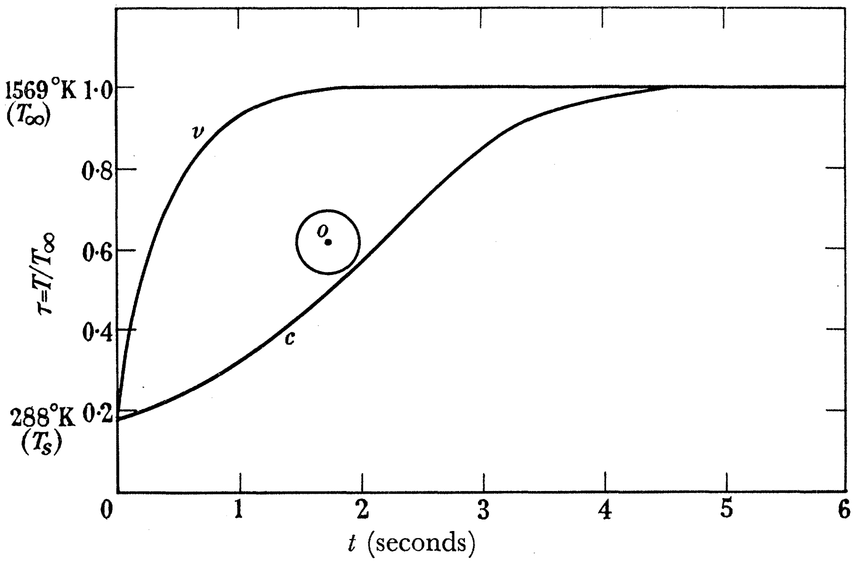

FIGURE 3. Heating-up curves for an infinite filament.

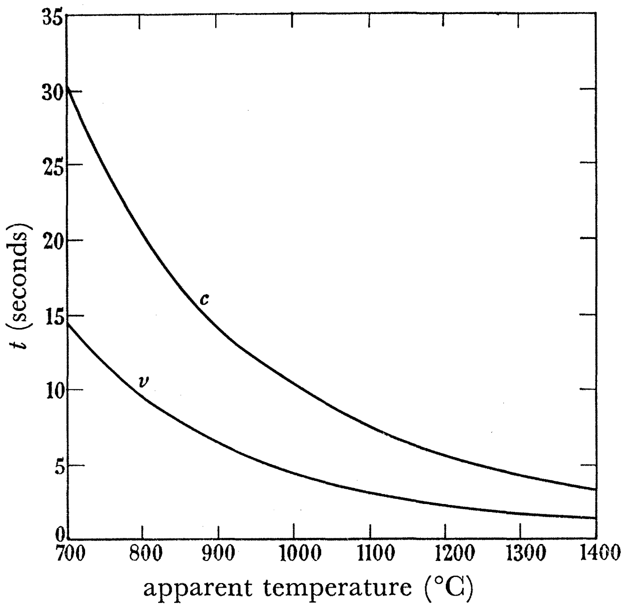

FIGURE 4. Time for an infinite filament to reach within 1 °C of maximum temperature.

so that its condition during heating-up fell between that of constant current and of constant voltage, but closer to the former owing to the substantial ballast resistance. A snap reading was taken as the filament reached visibility and this is shown in figure 3, the size of circle indicating the possible degree of error. It will be seen that this observation lies close to the *c* curve, and in the absence of disturbing conditions at the filament ends, the heating-up curve of the lamp might be expected to continue close to the *c* curve. If so, the time

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taken for the centre of the filament to approach within 1 °C of its final temperature would be given approximately by the *c* curve of figure 4. The *c* and *v* curves of the figure were calculated from (2·7·3) and (2·7·6), respectively. It will be seen that the times given by the *c* curve range from 30 s at 700° C to 3·4 s at 1400 °C.

However, the final approach to the steady state of the actual lamp is much more protracted. Thus at 1200 °C, instead of a time of the order of 5 s, our observations suggest one of some 10 min, and we note that the Code for disappearing-filament pyrometers issued by the British Standards Institution (1954) recommends a time of 30 min for steadying-up at 700 °C. The cause of this heavy creep and the possibility of eliminating it are discussed in the next section.

3·1·7. *Possible alterations in design*

The foregoing analysis of the characteristics of the standard strip lamp has revealed two sources of weakness, namely, the excessive creep and the heavy temperature coefficient of current calibration in the lower part of the scale.

The creep is almost certainly due to some factor or factors operative at the ends of the filament. One such factor might be the Peltier effect between tungsten and nickel, since its considerable variation with temperature would tend to slow down the attainment of equilibrium. If desired, this tendency could be eliminated by using for leads, instead of nickel, a material having a negligible thermal e.m.f. against tungsten;* such a material exists,† namely, the alloy 80 % Ni, 20 % Cr. But a more likely cause of creep is the conduction of heat from the filament to the mass of material of low diffusivity in the base of the lamp. Its effect might perhaps be minimized by interposing, between the sealing-in wires and the filament, a mass of metal considerably greater than the present leads, such metal being of high diffusivity and of shape and surface designed to give a high radiating power. The temperature change of the filament ends should then be small and quickly reached. Alternatively, the temperature of the ends could be strictly controlled by the use of water-cooled leads. Whether any such device would be worth considering would depend largely on the use for which the lamp is intended. Thus if habitually used for calibrating batches of pyrometers at a series of steady temperatures, the total time wasted by creep might represent an unimportant fraction in the life of the lamp; but the reverse might well be the case for a lamp used for short spells of observation or kept as an ultimate standard.

With regard to the heavy temperature coefficient of current calibration, some 30 % at 700 °C, this would be avoided by means of water-cooled leads. Otherwise the obvious remedy would be to use in this region the more favourable voltage calibration, with a temperature coefficient of only 3 % at 700 °C. To carry out this plan satisfactorily it would be advisable to provide separate potential leads to the filament ends. The voltage calibration could of course be used at higher temperatures and simultaneous readings of current and voltage would provide a valuable check on the constancy of the filament. Such a check could be made with ease and precision, but could not altogether replace the fundamental check against a standard black body, through an intermediary optical pyrometer, for this would cover changes from all causes, such as, for example, that due to blackening of the bulb.

* Such an elimination of the Peltier effect would be advantageous in any experiment concerning the Thomson effect in tungsten (see §3·1·5).

† See *Temperature*, Symposium of Amer. Inst. of Physics (1941), p. 1310.

It may be added that the system of water-cooled leads could readily be adapted to provide both current and potential leads. Thus if an electrode consisted of a continuous tube with a double seal through the glass, each half could serve one purpose, and similarly if an electrode consisted of two co-axial tubes (for inflow and outflow) which were insulated from each other except for a connexion at the filament end.

Finally, it is suggested that, if any substantial alteration in design is contemplated, consideration might well be given to the form of glass envelope to avoid the errors due to reflexion to which Barber (1946*a*) has drawn attention.

3·2. Pyrometer lamps

This section covers applications of our theory to pyrometer lamps in relation to their general characteristics in the steady state of temperature distribution (§3·2·1) and their approach to that state (§3·2·2).

3·2·1. *Characteristics in the steady state*

The majority of the formulae used for the standard strip lamp apply here and have been employed where appropriate. But there is an important difference between the two types. For the former, T_j is generally much higher than T_s (see table 5), whereas for pyrometer lamps the difference $T_j - T_s$ is often small enough to be negligible. If, as a typical example, we take a long flat filament, $0{\cdot}04 \times 0{\cdot}005$ mm in section, connected to nickel leads 0·5 mm diameter and 10 mm long, the formula (2·4·6) would make the difference 2 and 7 °C for true T_m temperatures of 1000 and 1400 °K, respectively. For a cylindrical filament of diameter 0·04 mm, presenting the same aspect as the other, the diameter of the leads would have to be increased to 0·86 mm to give the same result.

As explained in §2·5·3 the application of normalized variables is much simplified by the assumption $T_j = T_s =$ constant. It was shown that, for the same T_m, all lamps with the same dimensional relation $X = L(p/A)^{\frac{1}{2}}$ have identical voltage and temperature coefficients, as well as the same current relations $C = c/(Ap)^{\frac{1}{2}}$ and resistance relations $R = r_j(Ap)^{\frac{1}{2}}$. Table 10 gives calculations of the quantities mentioned, covering the normal range of temperature, and very roughly the limits of filament dimensions likely to be met in practice. With regard to the latter, the largest diameter ordinarily used is 0·065 mm which, in the past, has been combined with a length as short as 20 mm, giving a value of 24·8 cm$^{\frac{1}{2}}$ for X. Barber (private communication) found for such a filament a value of 1·39 for q (ratio of current Pd/Au) coupled with a current temperature coefficient at 700 °C (apparent) greater than 3. This last figure represents the possibility of an intolerably large error with change of air temperature and such lamps would not now be used. Accordingly we have chosen, somewhat arbitrarily, for the starting point of our table such a value for X as would reduce the coefficient at 700 °C to the (still considerable) value of 1·0. Lamps of this specification, with suitable leads, would give values of $T_j - T_s$ comparable with those in the preceding paragraph, which are close enough to the basic assumption.

In table 10, T_m is given as the true temperature (°K). To obtain the corresponding apparent temperature (°C) corrections would have to be applied for emissivity and for the absorption of the objective lens plus a single thickness of lamp bulb. The minimum absorption would be some 15 % (for four glass surfaces) and the maximum perhaps 20 %, depending on the type of objective. In practice the sole concern is with the properties of a lamp in relation to its apparent temperature when mounted in a particular pyrometer. The apparent temperatures given in the heading of table 10 are for a total absorption of 15 %, of which the objective lens accounts for 8 %. If the apparent temperatures are taken as standard, then for greater lens absorption small adjustments may be needed in the absolute values of the table (columns 3 to 6); the relative values (columns 2 and 7 to 9) would hardly be affected. Further, it may be pointed out that, though the figures in the table relate to a fixed T_s (15 °C), the normal range of air temperatures would have no appreciable affect even on the absolute values.

TABLE 10. CURRENT, RESISTANCE AND VOLTAGE CALIBRATIONS, AND TEMPERATURE COEFFICIENTS, OF PYROMETER LAMPS, CORRELATED WITH THE QUANTITY $X = L(p/A)^{\frac{1}{2}}$ AND WITH q THE Pd/Au RATIO OF CURRENT

$C = c/(Ap)^{\frac{1}{2}}$, $R = r_j(Ap)^{\frac{1}{2}}$, $V = v_j$; L, R, V, refer to *half* the filament.
Assumed $T_j = T_s = 288$ °K.
T_m in true temperature (°K); corresponding apparent temperatures (°C) for minimum absorption (15 %, see above) are:

1000 °K (700 °C), 1390 °K (1063 °C = Au), 1931 °K (1552 °C = Pd).

X	q	T_m	C	$10^5(R/X)$	V	$(\mathrm{d}T_m/\mathrm{d}T_s)$ for constant		
(cm$^{\frac{1}{2}}$)	$c_{\mathrm{Pd}}/c_{\mathrm{Au}}$	(°K)	(A cm$^{-\frac{3}{2}}$)	(Ω cm)	(V)	amp	ohm	volt
33·67	1·625	1000	285·8	1·948	0·1875	+0·98	−0·54	+0·08
		1390	371·5	2·881	0·3604	+0·25	−0·46	−0·09
		1931	603·7	4·598	0·9345	$+0{\cdot}01_9$	−0·28	−0·11
40·25	1·729	1000	250·3	1·958	0·1972	+0·73	−0·53	+0·03
		1390	346·1	2·934	0·4088	+0·15	−0·42	−0·11
		1931	598·4	4·750	1·1440	$+0{\cdot}00_8$	−0·25	−0·09
44·27	1·777	1000	234·5	1·964	0·2039	+0·62	−0·52	+0·01
		1390	336·1	2·970	0·4419	+0·11	−0·39	−0·11
		1931	597·3	4·820	1·2746	$+0{\cdot}00_6$	−0·23	−0·09
∞*	1·914	1000	159·0	2·570	—	$+0{\cdot}02_4$	0	$+0{\cdot}01_6$
		1390	311·5	3·819	—	$+0{\cdot}00_9$	0	$+0{\cdot}00_6$
		1931	596·3	5·666	—	$+0{\cdot}00_3$	0	$+0{\cdot}00_2$

* Cf. table I of Langmuir (1916*c* or *b*), applicable to very long filaments; Langmuir's $2A'/\pi$ and $\pi R'/4$ can be compared with our C and R/X in the case $X = \infty$. In this case $R/X = \rho$, the resistivity at the temperature in question.

Very little experimental evidence can be adduced in direct support of table 10 especially in the important matter of temperature coefficients. However, Barber (private communication) found that a lamp with a value of $1{\cdot}62_5$ for q had the following temperature coefficients of current calibration for the apparent temperatures given in brackets:

1·0 (700 °C), 0·2 (1063 °C), 0·0 (1552 °C).

It will be seen that these agree with the figures in the table, certainly to the nearest 0·1, beyond which the figures have no practical significance. A comparison between calculated and observed values is also possible for the total voltage across the filament. For the three temperatures named, the calculated voltages from table 10 are shown in brackets after Barber's observed values, namely, 0·384 (0·375), 0·736 (0·721), 1·920 (1·869). It will be seen that the differences between pairs of values all come within 3 %. The further comparison with columns 4 and 5 of the table was rendered impossible by some uncertainty in the filament dimensions. In the case of another filament, of dimensions $0{\cdot}04\times0{\cdot}005\times25$ mm (so that $X=83{\cdot}9$), Barber found a negligible current coefficient at 800 °C (apparent) coupled with a value of 1·90 for q, which figures are consistent with the table.

In view of these facts and the general endorsement of our theory in other directions, it would seem that the table can be used with some confidence to obtain approximate values of the main characteristics of pyrometer lamps at the three temperatures named and by interpolation between these temperatures. Thus, for any particular dimensions of filament, a lamp designer could get the value of X and hence, from columns 4 to 6, the current, resistance and voltage over the desired range of temperature. These data fix the voltage and capacity of the battery required for the circuit and the sensitivity demanded of the measuring appliance. Then from columns 7 to 9 can be obtained an estimate of the reproducibility, with varying air temperature, of the scale of calibration. The choice of the calibration characteristic depends somewhat on the range to be covered, but it will be seen from the table that, for a tolerable limit of 0·1, the one for voltage is satisfactory throughout. The current relation shows a tendency to large errors at low temperatures but quickly improves with rising temperature. The resistance relation is liable to rather high errors throughout, and shows little improvement for rising values of X. This last fact suggests that relatively long filaments would have to be used to reduce the coefficient to reasonable figures—in fact, we calculate that the value of $X=L(p/A)^{\frac{1}{2}}$ required at 700 °C (apparent) in order to reduce the resistance coefficient $(\mathrm{d}T_m/\mathrm{d}T_s)_r$ to $-0{\cdot}1$ is about 390 $\mathrm{cm}^{\frac{1}{2}}$; and a filament of cross-section $0{\cdot}04\times0{\cdot}005$ mm would have to be 116 mm long in order to achieve this. It may, however, be possible with a bridge circuit to reduce or eliminate the coefficient by having in an arm, opposed to the lamp filament, a resistance of appropriate coefficient and dimensions. Such resistance would best be in the form of a short loop of tungsten similar to the filament and be included in the bulb so as to be exposed to the same temperature changes and show similar lag. For adjustment, there would be available the relative dimensions of loop and filament and the proportions of current carried by the two.

In addition to the use of the table in predicting the behaviour of hypothetical lamps, it can be employed with advantage for lamps already in existence without the necessity of determining the filament dimensions. Thus for any T_m the current, resistance and voltage can readily be measured by the ordinary method of calibration against a standard, and with q so determined the temperature coefficients could be estimated from the table. In this way the troublesome process of actually measuring any of the coefficients could be obviated.

Finally, it may be pointed out that the table serves to illustrate some of the advantages of a filament with flat, as against one with circular, cross-section. Taking, for example, flat ($0{\cdot}04\times0{\cdot}005$ mm) and cylindrical (0·04 mm in diameter) filaments (of identical aspect), the latter would have, for the same X, a length 2·12 times that of the former, and values of current and conductance (reciprocal of resistance) 2·96 times those of the former. The first-mentioned feature would tend to the advantage of the flat filament by simplifying design and reducing the risk of vibration, especially with the filament mounted as an edgewise loop; while the second feature would allow the use of a smaller battery. It should be added that, with its higher watts, the sensitivity given by the cylindrical filament would be the greater, but this may be unimportant if that of the flat filament is adequate. The above-mentioned advantages are in addition to the quicker response of the flat filament (see §3·2·2 below) and to its outstanding merits, in relation to the optics of pyrometry, established by Fairchild & Hoover (1923).

3·2·2. *Speed of response*

In this section we compare the results of calculation of speed of response of pyrometer lamps with those given by Barber (1946*b*). For this purpose we reproduce curves from his figures 1 and 3 on our figure 5. The curves show the time taken, from switching on, for the filament centre to reach within 2 °C of its final value, which ranges from 800 to 1200 °C apparent. The filaments were of two diameters, 0·06 and 0·04 mm, the latter being used in

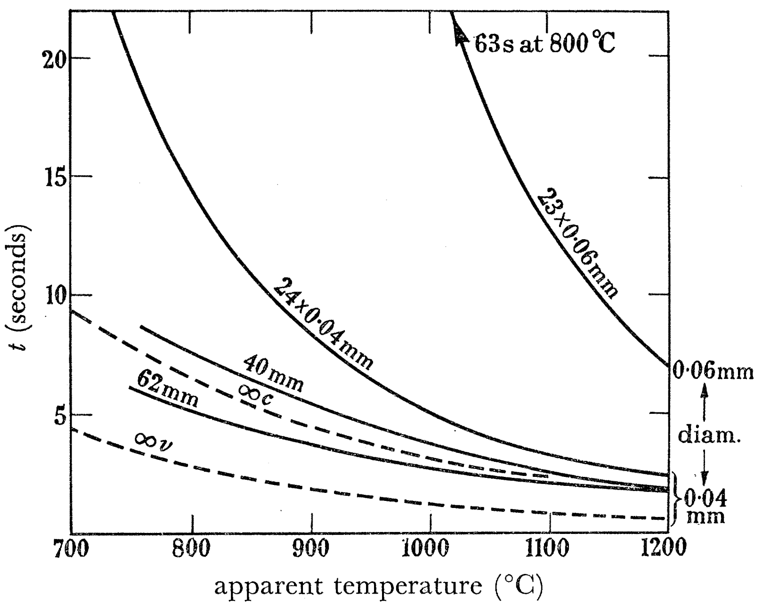

FIGURE 5. Times taken for various pyrometer filaments to reach within 2 °C of final temperature.

three lengths. Our calculated curves apply to an infinitely long filament of diameter 0·04 mm, run at constant current and voltage, respectively, and are shown as dashed lines marked ∞*c* and ∞*v*. As already pointed out in §2·7, if a long filament is run with some ballast resistance in series, its response curve would lie between the *c* and *v* ones, but probably closer to the former. As considerable ballast was no doubt present in Barber's circuits and his curve for the 0·04 × 62 mm filament falls between our two curves, we may conclude that this length gives a response indistinguishable from that for an infinite filament. For shorter lengths, of the same diameter, the response times are seen to be greater, the value at 800 °C for the 24 mm filament being 2·3 times that for our ∞*c* curve.

The top curve of figure 5 relates to a filament 0·06 mm in diameter, which is about the largest employed in pyrometers. The length (23 mm) is relatively so short that it would not now be used on account of its heavy temperature coefficients, as explained in the preceding section. It will be seen that its response time for 800 °C is some 60 s which is about seven times that for its ∞*c* curve.

It is also of interest to compare our times for a cylindrical filament, 0·04 mm in diameter, with those for a corresponding flat filament, for example, that treated in the remarks following table 10, of cross-section $0{\cdot}04 \times 0{\cdot}005$ mm. As shown in §2·7·2 and §2·7·3 above, the response time is proportional to A/p, which would make the flat filament about 4·5 times as rapid in response as the cylindrical one.

In conclusion, we desire to acknowledge our great indebtedness to the Director of the National Physical Laboratory for the calculations undertaken by the Mathematics Division and the data supplied by the Physics Division. Our thanks are also due to the referees for suggestions which have led to several clarifications.

Appendix. The theory of Ribaud & Nikitine (1927) for a short-filament lamp

In our notation, the basic equation for the steady state of heat flow in the filament is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak_T\frac{\mathrm{d}T}{\mathrm{d}x}\right) = \sigma p\epsilon_T(T^4 - T_s^4) - \frac{c^2\rho_T}{A}. \tag{A 1}$$

First, define a temperature T_∞ which would be the temperature maintained by the current c in an infinite filament of the same material and cross-section as the finite filament, so that if ρ_∞, ϵ_∞ are the values of ρ_T, ϵ_T at $T = T_\infty$, then

$$c^2\rho_\infty = A\sigma p\epsilon_\infty(T_\infty^4 - T_s^4). \tag{A 2}$$

If $\rho_T = B'T^\beta$ (B' = constant), a linear approximation near $T = T_\infty$ is

$$(\rho_T - \rho_\infty)/\rho_\infty = \beta_\infty(T - T_\infty)/T_\infty; \tag{A 3}$$

there is a similar linear approximation if $\sigma\epsilon_T T^4 = A'T^\gamma$ (A' = constant), and if also $k_T = k$ (constant), the differential equation (A 1) takes the simple form

$$\mathrm{d}^2T/\mathrm{d}x^2 = P^2(T - T_\infty),$$

where $$P^2 = (\sigma p\epsilon_\infty/Ak)\,[(\gamma_\infty - \beta_\infty)\,T_\infty^3 + (4 + \beta_\infty - \gamma_\infty)\,T_s^4/T_\infty].$$

(In the original treatment, $T_s = 0$ and P^2 is correspondingly simplified; in practice, the term in P^2 involving T_s is negligible compared with the other term—in fact, on our assumptions for tungsten, $\gamma_\infty = 5$, $\beta_\infty = 1$ and so $4 + \beta_\infty - \gamma_\infty = 0$.) With the origin at the maximum ($x = 0$ when $T = T_m$) we then obtain

$$\frac{T_\infty - T}{T_\infty - T_m} = \cosh Px. \tag{A 4}$$

This formula corresponds with our elliptic form (2·1·5). To find a c, T_m relation in the symmetrical case, the simplest method is to use T_∞ as a parameter, calculating c from (A 2) and T_m from (A 4) with $T = T_j$ (or T_s) and $x = L$ (the semi-length of the filament).

It is easy to extend this theory to cover the calculation of resistance or volts. Thus the resistance of a section of the filament measured from $T = T_m$ is, by (A 3) and (A 4),

$$r = \frac{1}{A}\int_0^x \rho_T\,\mathrm{d}x = \frac{\rho_\infty}{AT_\infty}\int_0^x [T_\infty + \beta_\infty(T - T_\infty)]\,\mathrm{d}x$$

$$= \frac{\rho_\infty}{AT_\infty}\int_0^x [T_\infty - \beta_\infty(T_\infty - T_m)\cosh Px]\,\mathrm{d}x;$$

i.e. $$r = \frac{\rho_\infty x}{A} - \frac{\rho_\infty\beta_\infty(T_\infty - T_m)}{APT_\infty}\sinh Px. \tag{A 5}$$

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